B.Sc. STATISTICS - I YEAR

DJS1A : REAL ANALYSIS AND MATRICES

SYLLABUS

Unit - I

Sets - Countability, Open and Closed sets of real numbers. Sequences – Convergent and Divergent sequences, Bounded and Monotone sequences, Cauchy sequences. Series of real numbers - Convergence and divergence-series with nonnegative terms - comparison test - D'Alembert's ratio test - Cauchy's root test. - conditional and absolute convergence.

Unit - II

Differentiation **-** Limit of a function of a single variable, Continuity properties of a continuous function in a closed interval, Derivatives, Rolle's Theorem, Mean value theorem, Taylor's theorem.

Unit - III

Integration - Concept of Riemann Integral, Sufficient condition for Riemann integrability, Darboux theorem, Fundamental theorem, First mean value theorem – Improper Riemann integrals. Beta and Gamma Integrals.

Unit - IV

Matrices - Operations on Matrices – Symmetric and Skew-symmetric Matrices – Conjugate of a Matrix – Determinant of a Matrix – Inverse of a Matrix. Solving system of linear equations. Elementary transformations, Elementary matrices, Row and Column ranks – rank of a matrix. Reduction to Normal form, Equivalent matrices.

Unit - V

Characteristic roots and vectors, Cayley- Hamilton theorem, Minimal equation of a matrix. Quadratic Form – Matrix of a quadratic form – rank, signature and classification of quadratic forms – Sylvester's of Inertia.

REFERENCE BOOKS::

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Unit - I

1.1 Sets:

A set is a collection of objects (or) elements. Typically, the type of all the elements in a set is the same. For example - All the elements in a set could be integers. However, it is possible to have different types of elements in a set. (An analogy for this is that usually a book bag contains just books. But sometimes it may contain other elements such as pencils and folders as well). We have two usual methods of denoting the elements in a set:

1) Explicitly list all the elements inside a set of curly braces {}, as follows: {1,2,4,5,6,7}

2) Given a description of the elements in a set inside of a set of curly braces as follows: $\{2x \mid x \in N\}$.

To understand the second method, we must define the various symbols that are used in this notation. Here is a list of the symbols we will be using:

| - translates to "such that"
$\in$- "is an element of"
$\subset$- "is a proper subset of"
$\subseteq$- "is a subset of"

Now we must define what a subset is. A subset is also a set. So, if we have sets A and B, $A \subseteq B$ if for all $x \in A$, $x \in B$. In layman's terms, a set A is a subset of a set B, if all the elements in the set A also lie in the set B.

Note: $A \subset B$ iff $A \subseteq B \wedge A \neq B$.

We still must define what $\{2x \mid x \in N\}$ really means. Here it is in English: "The set of all numbers of the form 2x such that x is an element of the natural numbers." (Note: The set N denotes the natural numbers, or the non-negative integers as per the book). So, the set above could also be listed as {0, 2, 4, 6, ...}. Now that we have gotten that out of the way, let's talk about the empty set ($\varnothing$). The empty set is a set with no elements in it. In our standard notation, we could denote it as {}. It is also very common to use $\varnothing$, to denote the empty set. It's important to denote that the following are not equal: $\varnothing$, {0}, and 0. The first two are sets, while the third is an element. However, the empty set has no elements while {0} contains one element, zero. Typically, sets will be denoted by uppercase letters. There are some other sets we should be familiar with since they come up so often. Here they are:

Z = {0, 1, -1, 2, -2, ...} (the set of integers)
N = {0, 1, 2, 3, ...} (the set of non-negative integers)
Z^+ = {1, 2, 3, ...} (the set of positive integers)
Q = {a/b | a, b$\in$Z $\wedge$ b$\neq$0} **R = the set of real numbers...**

Also, one last definition... |A| for a set A is known as the "cardinality" of A, which equals the number of elements in A.

1.2 SET OPERATORS:

Now we are ready to discuss set operators. We can use several operators on existing sets to define new ones. The first two operators are binary operators, union and intersection. In each of these examples, let A and B be sets.

Union ($\cup$): $A \cup B = \{x \mid x\in A \vee x\in B\}$ **Intersection ($\cap$): $A \cap B = \{x \mid x\in A \wedge x\in B\}$** **Complement ($\neg$):$\neg A = \{x \mid x\notin A\}$** **Relative complement (–): $B - A = \{x \mid x\in B \wedge x\notin A\}$**

In General, the union of two sets contains all elements in either set and the intersection of two sets contains all elements in both sets. To define the complement, we must define what a universe is. For each set, there is a possible set of elements. This possible set of elements is known as the universe. Typically, you will be told what the universe is for each problem.

The complement of a set contains all the elements in the universe that are NOT in the set itself. You can think of relative complement as the subtraction between two sets. B – A refers to a set that subtracts out all the elements from A out of B. Now if an element of A wasn't in B to begin with, there's no need to take it out of B at all... Also, an identity that we can use is that $B - A = B \cap \neg A$.

1.3 Equality of Sets:

There are three different ways that we can show two sets to be equal. The first two are going to be analogous to the methods used in logic.

1) Use the laws of set theory.

2) Use the table method.

Use the laws of set theory

1. $\neg\neg A = A$	**Law of Double Complement**
2. $\neg(A \cup B) = \neg A \cap \neg B$	**De Morgan's Laws**
$\neg(A \cap B) = \neg A \cup \neg B$	**De Morgan's Laws**
3. $A \cup B = B \cup A$	**Commutative Laws**
$A \cap B = B \cap A$	**Commutative Laws**
4. $A \cup (B \cup C) = (A \cup B) \cup C$	**Associative Laws**
$A \cap (B \cap C) = (A \cap B) \cap C$	**Associative Laws**
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	**Distributive Laws**
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	**Distributive Laws**
6. $A \cup A = A$, $A \cap A = A$	**Idempotent Laws**
7. $A \cup \varnothing = A$, $A \cap U = A$	**Identity Laws**
8. $A \cup \neg A = U$, $A \cap \neg A = \varnothing$	**Inverse Laws**
9. $A \cup U = U$, $A \cap \varnothing = \varnothing$	**Domination Laws**
10. $A \cup (A \cap B) = A$	**Absorption Laws**
11. $A \cap (A \cup B) = A$	**Absorption Laws**

1.4 Countability

1.4.1 Countable and Uncountable sets:

Two sets A and B are said to be equivalent if there exists a function f: $A \rightarrow B$, which is one – to - one and onto. If A is equivalent to B, we write $A \sim B$

Examples: -

1. $\{a, b\} \sim \{1,2\}$
2. $\{a,b,c\} \sim \{x,y,z\}$
3. $\{1,2,3 \ldots\ldots\} \sim \{2,4,6 \ldots\ldots\}$

1.4.2 Definition:

A set S is said to be countable (or denumerable) if either S is finite or S is equivalent to N, the set of all positive integers. An infinite set which is not countable, is said to be uncountable (or non-denumerable)

Examples: -

1.The set {A, B, C} of the vertices of a triangle is countable, it is a finite set, and hence countable.

2. The empty set $\emptyset$ is countable; it is a finite set and hence countable.

3. The set n of all positive integers is countable, the identity function I: N→N is one -to -one onto, and hence N is countable.

4. The set Z of all integers is countable. Define f: N→Z, by the rule f(n)=$\frac{n-1}{2}$, n=1,3, 5…. and f(n)=$\frac{-n}{2}$, n=2, 4…. Then f: N→Z, is one- to -one onto. Therefore Z~N, and hence Z is countable.

5. Show that the set R of all real numbers is uncountable.

We know, the set [0,1] is uncountable, and since [0,1] $\subseteq$ R therefore the set R is also countable

1.4.3 Theorem:

If a set A is countable, and B $\subset$ A, then B is also countable.

Proof: Since A is countable, there exists a function f: N→A, which is one –to- one onto. Let f(n)=a_n, n$\in A$ n=1,2,3,4, 5…….. the elements of a can be arranged as $a_1, a_2. a_3, a_4 \dots \dots \dots a_n$.

Now, we define a function g: N→B, by the rule: Let n_1 be the first positive integer, such that $a\ _{n_1} \in B$.Set g (1) =$a_{n_1}, a_{n_1} \in B$. Let n_2>n_1 be the next positive integer, such that $a_{n_2} \in B$.

Set g (2) =a_{n_2}. Continuing in this way, we get g(k) =$a_{n_k}.a_{n_k} \in B. n_k$>$n_{k-1}$, for all k $\in$N, then clearly g: N→ B is one- to- one and onto. Therefore B ~N, and hence, B is countable

1.4.4 Example:

The set of all rational numbers in [0, 1] is countable.

Solution: We know that, the set Q^* of all positive rational numbers is countable

Now, {all rational numbers in [0, 1]} =Q^* ∩[0,1] $\subset Q^*$.

Therefore, the set of all rational numbers in [0,1] is countable.

1.4.5 Theorem:

The set of irrational number is uncountable.

Proof: Let A denote the set of irrational numbers. Let, if possible, A is countable. We know that the set Q of rational number is countable.

Since A and Q are uncountable A$\cup$ Q, i.e, R must be countable. But R is not countable. Thus, the assumption that A is countable leads to contradiction. Hence A, i.e., the set of irrational numbers is uncountable.

1.4.6 Open and Closed sets of real numbers:

A set is said to be open if it is a neighbourhood of each of its points. Thus, if A be an open set and x is any member of A, then by the definition of an open set an open interval] a, b[such that xϵ]a, b[$\subset$A. Equivalently, A is open if for each xϵA, there exists $\in$>0 such that]x- $\in$, x+ $\in$[$\subset$ A.

Note: To show that A is not open we should prove that there exists at least one point of A of which is not a neighbourhood i.e. there exists some xϵ A such that for each $\in$>0, however small] x- $\in$, x+ $\in$ [is not a sub-set of A.

1.4.7 Definition:

A set G $\subseteq$ R is said to be an open set, if it is a neighbourhood of each of its points.

1.4.8 Theorem:

A set G $\subseteq$ R is open if and only if, for each p$\in$G, there exists a δ >0 such that $^{N}_{b}(p)\subseteq$ G.

Proof:

i) The condition is necessary. Let G $\subseteq$ R be an open set and let p be any point of G. By definition G is a neighbourhood of p, $\therefore$ $\exists$ b>0, such that$^{N}_{b}(p)\subseteq$ G.

ii) The condition is also sufficient. Let G $\subseteq$ R and suppose for each p$\in$ $G\exists$ b>0, such that$^{N}_{b}(p)\subseteq$G. Then, for each p$\in$ G is a neighbourhood of each of its points. $\therefore$ G is open.

1.5 Sequences:

A sequence is a set function of domain is the set N of natural numbers whereas the range may be to set S. In others words a sequence in a set S in a rule which assigns to each natural numbers a unique element of S.

The elements of the set can be either numbers or letters or a combination of both. The elements of the set all follow the same rule (logical progression). The number of elements in the set can be either finite or infinite. A sequence is usually represented by using brackets of the form {} and placing either the rule or a number of elements inside the brackets. Some simple examples of sequences are listed below.

The alphabet: {a, b, c, ..., z}, The set of natural numbers less than or equal to 50: {1, 2, 3, 4, ..., 50}, The set of all natural numbers: {1, 2, 3, ..., n, ...}, The set $\{a_n\}$ where $a_n = a_{n-1} + 1$, $a_1 = 1$.

1.5.1 Real sequence:

A real sequence is a function where domain is the set N of all natural numbers and range a subset of the set R of real numbers symbolically f: N→R or x: N→R is a real sequence.

The sequence is denoted by $\{x_n\}$ or > x < where $x_1, x_2, \ldots\ldots\ldots.. x_n$ are called 1,2……n terms of the sequence and it occurs different position and are treated as distinct terms.

1.5.2 Range of Sequence:

The set of all distinct term of a sequence is called its range. In a sequence since f∈N is an infinite set N. The range of a sequence may be a finite set. For example: If $x_n = (-1)^n$, then x_n= {-1, +1,-1,+1…………} the range = {-1, +1}

1.5.3 Constant sequence:

A Sequence $\{x_n\}$ defined by x_n =C ∈R $\forall\, n \in N$ is called a constant sequence. Thus $\{x_n\}$={c,c,c………c} is a sequence with range {c}.

1.5.4 Algebra of sequences:

Given any two sequences $\{a_n\}$ with limit value A, $\{b_n\}$ with limit value B, and any two scalars k, p, the following are always true:

(a) $\{k\, a_n + p\, b_n\}$ is a convergent sequence with limit value kA + p B.

(b) $\{a_n * b_n\}$ is a convergent sequence with limit value AB.

(c) $\{\frac{a_n}{b_n}\}$ is a convergent sequence with limit value A/B provided that B ≠ 0.

(d) if f(x) is a continuous function with $\lim_{x\to\infty} f(x) = L$, and if $a_n = f(n)$ for all values of n then $\{a_n\}$ converges and has the limit value L.

(e) if $a_n \le c_n \le b_n$, then $\{c_n\}$ converges with limit value C where $A \le C \le B$.

Note 1:

If each element of a sequence $\{a_n\}$ is no less than all its predecessors ($a_1 \le a_2 \le a_3 \le a_4 \le \ldots$) then the sequence is called an increasing sequence. If each element of a sequence $\{a_n\}$ is no greater than all its predecessors ($a_1 \ge a_2 \ge a_3 \ge a_4 \ge \ldots$) then the sequence is called a decreasing sequence.

Note 2:

A monotonic sequence is one in which the elements are either increasing or decreasing. If there exists a number M such that $|a_n| \le M$ for all values of n then the sequence is said to be bounded.

1.5.5 Convergent Sequence:

If $\lim_{n\to\infty} a_n = l$ then the sequence a_n converge to *l*.

Equivalently a sequence a_n is said to converge to a real number *l* (i.e) if given $\varepsilon > 0$, however small, $\exists$ a positive integer m such that $|a_n - l| < \varepsilon \forall\, n \ge m$ the real number *l* is called the limit of the sequence $\{a_n\}$.

1.5.6 Divergent Sequence:

1. A sequence a_n is said to be divergent to ∞ for any positive real number k between large when $\exists$ a positive integer m such that

$$a_n > k \,\forall\, n \ge m \;(i.e)\; \lim_{n\to\infty} a_n \ne l, \lim_{n\to\infty} a_n = \infty / \, a_n \to \infty \; as\; n \to \infty$$

2.A sequence a_n is said to divergent to $-\infty$ for any positive real number k however large then $\exists$ a positive integer m such that

$$a_n < -k \,\forall\, n \ge m \;, \lim_{n\to\infty} a_n = -\infty / \, a_n \to -\infty \; as\; n \to \infty$$

3. A sequence a_n is said to be a divergent sequence if it diverges to ∞ or $-\infty$ (i.e) $a_n = \infty$ or $a_n = -\infty \; as\; n \to \infty$

Example: the sequence is {n} and $\{n^2\}$ diverge to $+\infty$ similarly the sequence –n and –n^2 diverge to -∞.

1.5.7 Standard Sequences:

Some of the most important sequences are

(1) $\{r^n\} = \{r^1, r^2, r^3, \cdots\}$. This sequence converges whenever -1 <$r \le$ 1.

(2) $\{n^r\} = \{1^r, 2^r, 3^r, \cdots\}$. This sequence converges whenever r $\le$ 0.

1.6 Bounded and unbounded sequence

1.6.1 Bounded Sequence:

A Sequence is said to be bounded if it is bounded above as well as below. Thus, the sequence a_n *is bounded if there exist* two real numbers k and K $\ni$ k$\leq$ $a_n \leq K \forall\, n \in N$ where {k $\leq K\}$ (i.e) if the range of the sequence is bounded. A Sequence is said to be unbounded if it is not bounded.

1.6.2 Bounded Above Sequence:

A Sequence $\{a_n\}$ is said to be bounded above if there $\exists$ a real number k such that $a_n \leq k \forall\, n \in N$ (i.e) if the range of the sequence is bounded above.

1.6.3 Bounded Below Sequence:

A Sequence $\{a_n\}$ is said to be bounded above if there $\ni$ a real number k such that $a_n \geq k \forall\, n \in N$ (i.e) if the range of the sequence is bounded below.

1.6.4 Least Upper Bound of a Sequence:

If a Sequence $\{a_n\}$ is said to be bounded above if there $\ni$ a real number such that $a_n \leq k_1 \forall\, n \in N k_1$ is called upper bound of the sequence. If $k_1 \leq k_2$ then $a_n \leq k_2 \forall\, n \in N$,$k_2$ is the bound of the sequence implies, if any number >k_1 *is also upper* bound of the sequence. Therefore, if a sequence is bounded above it has infinitely many upper bounds of all upper bounds of the sequence, if k is the least then k is called a least upper bound (LUB) of the sequence. It has the following properties. It is an upper bound of the sequence $a_n \leq k \forall\, n \in N$ given ε >0, k- ε, k.Since k is the (LUB), k- ε *is not* even an upper bound. Implies there exists at least one positive integer m such that a_m, Not less than are equal to k- ε, *Implies* $a_m \geq$ k- ε.

1.6.5 Greatest Lower Bound of a Sequence:

If a sequence $\{a_n\}$ is bounded below then $\exists$ a real number such that

$k_1 \leq a_n \forall\, n \in N, k_1$ is called lower bounded of the sequence. If $k_2 < k_1$, then $k_2 \leq a_n \forall\, n \in N$, implies k_2 is also a lower bound of the sequence, if a sequence is bounded below. If a sequence $\{a_n\}$ is bounded below then infinitely many lower bound of all the lower bounds of the sequence. If k is the greatest, then k is called greatest lower bound (GLB) of the sequence.

It has the following properties:

1. It is the lower bound of the sequence implies $k > a_n \forall\, n \in N$.

2. Given $\varepsilon > 0, k + \varepsilon > k$ since is the greatest lower bound (GLB) $k + \varepsilon$ is not even a lower bound implies at least one positive integer such that k+k not less than or equal to a_n implies $k + k > a_n \forall\, n \in N$ (or) $k + k > a_n \forall\, n \in N$.

<u>1.6.6 Limit of a Sequence</u>:

Let $\{a_n\}$ be a sequence and $l \in R$. The real number l is said to be the limit of a sequence $\{a_n\}$ if to each $\varepsilon > 0, \exists\, m \in N$, (m depending on ε) such that $|a_n - l| < \varepsilon \forall\, n \geq m$ ∴The limit of $\{a_n\}$ then

$\{a_n\} \rightarrow l \; as\, n \rightarrow \infty \; (or) \; \lim\limits_{n \to \infty} a_n = l$.

<u>1.6.7 Monotone Sequence:</u>

1. A sequence $\{a_n\}$ is said to be monotonically increasing, if

$\{a_{n+1}\} \geq \{a_n\} \forall\, n \in N$

(i.e) $a_1 \leq a_2 \leq a_3 \leq a_4$…….$\leq a_n$

2. A sequence $\{a_n\}$ is said to be monotonically decreasing,

if $\{a_{n+1}\} \leq \{a_n\} \forall\, n \in N$

(i.e) $a_1 \geq a_2 \geq a_3 \geq a_4$…….$\geq a_n$

3. A sequence $\{a_n\}$ is said to be monotonic. If it is either monotonically increasing or decreasing.

4. A sequence $\{a_n\}$ is said to be strictly monotonic increasing.

$\{a_{n+1}\} > \{a_n\} \forall\, n \in N$

5 A sequence $\{a_n\}$ is said to be strictly monotonic decreasing.

$\{a_{n+1}\} < \{a_n\} \forall\, n \in N$

6. A sequence $\{a_n\}$ is said to be strictly monotonic. If it is either strictly monotonically increasing or strictly monotonically increasing.

1.6.8 Theorem:

Every convergence sequence has a unique limit or a sequence cannot converge to more than one limit.

Proof: Assuming the continuity $\lim S_n = l$ and $\lim S_n$=m where $l \neq m$

Then $|l - m| \geq 0$. Let $\epsilon = |l - m|$, since

$\lim S_n = l\ \exists M_1 \in N\ such that\ |S_n - m| \leq \frac{\epsilon}{2}, \forall n \geq M_1$ …………….………. (1)

Similarly, since $\lim S_n = m\ \exists M_2 \in N\ such that\ |S_n - m| \leq \frac{\epsilon}{2}, \forall n \geq M_2$ …… (2)

Let M = max (M_1, M_2), then (1) & (2) holds the $\forall n \geq M$ we have

$|l - m| = |l - S_n + S_n - m| < |l - S_n| + |S_n - m| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$

[from (1) & (2)] $= \epsilon = |l - m|$

(i.e) l=m.

1.7 Cauchy general Principle of Convergence of a Sequence:

1.7.1 Theorem:

The necessary and sufficient condition for the convergence of the sequence a_n is that to every positive number ϵ, however small their correspondence a positive integer m such that $|a_{n-p} - a_n| < \epsilon \forall n \geq M$ and for all integer values of p>0.

Proof:

1. Necessary Condition

Let the sequence be convergent that is, it has a finite limit say A when given $\in$ however small $\exists$ a positive integer m such that $|a_n - A| < \frac{1}{2}\epsilon \forall n \geq M$, it follows that

$$|a_{n+p} - A| = |a_{n+p} - A + A - a_n| \leq |a_{n+p} - A| + |A - a_n| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon\ \forall n \geq M.$$

2. Sufficient Condition

Let $|a_{n+p} - a_n| < \epsilon \forall n \geq M$ is p>0 taking n=m we get

$|a_{m+p} - a_n| < \epsilon \forall n \geq M$ is p>0. Since $a_n$is finite it follows the a_{m+p} different from a_m which $<\epsilon$ however large p may be (i.e.) $\lim_{n\to\infty} a_{m+p}$ is finite, $\lim_{n\to\infty} a_m$ is finite moreover since $|a_{m+p} - a_n| < \epsilon$ it follows that $\lim_{n\to\infty} a_{m+p}$ cannot be different from $\lim_{n\to\infty} a_m$ (i.e) the sequence has a unique limit. Hence it is convergent thus the conditional is sufficient.

Example: 1:

Apply the Cauchy principle of convergence to show that the series $1+\frac{1}{2} + \frac{1}{3} + \cdots \ldots \frac{1}{n}$is not convergent

Solution: $S_n=1+\frac{1}{2}+\frac{1}{3}+\cdots\ldots\frac{1}{n}$,$S_{n+p}=1+\frac{1}{2}+\frac{1}{3}+\cdots\ldots\frac{1}{n}+\ldots\ldots\frac{1}{n+p}$ suppose that series converges $\exists$ a positive integer m such that for every n$\geq$ m and for every p>0, we have $\left|S_{n+p}-S_n\right|<\epsilon$ where ϵ is an arbitrary small quantity (i.e)

$\left|S_{n+p}-S_n\right|<\epsilon=\left|1+\frac{1}{2}+\frac{1}{3}+\cdots\ldots\frac{1}{n}+\cdots\ldots.\frac{1}{n+p}-1+\frac{1}{2}+\frac{1}{3}+\cdots\ldots\frac{1}{n}\right|$ as implies

$\left|\frac{1}{n+1}+\frac{1}{n+2}+\cdots\ldots.\frac{1}{n+p}\right|<\epsilon$ in particular when n=m and p =m we see that

$\frac{1}{m+1}+\frac{1}{m+1}+\cdots\ldots.\frac{1}{m+m}>\frac{m}{2m}=\frac{1}{2}$ Now ϵ is at our choice and taken the values containing $<\frac{1}{2}$. Thus, it is contradiction. Hence the series is not convergent.

Example: 2

Verify the series is convergent are not $S_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ldots\ldots\frac{(-1)^{n-1}}{n}$

Given $S_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ldots\ldots\frac{(-1)^{n-1}}{n}$,$S_{n+p}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ldots\ldots\frac{(-1)^{n-1}}{n}+\ldots\ldots\frac{(-1)^{n+p-1}}{n+p}$

Therefore,

$\left|S_{n+p}-S_n\right|=$

$\left|1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ldots\ldots\ldots\frac{(-1)^{n-1}}{n}-1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ldots\ldots\ldots\frac{(-1)^{n-1}}{n}+\cdots\frac{(-1)^{n+p-1}}{n+p}\right|$

$=\left|\frac{(-1)^{n+1-1}}{n+1}+\frac{(-1)^{n+2-1}}{n+2}\frac{(-1)^{n+3-1}}{n+3}+\cdots\ldots\ldots\frac{(-1)^{n+p-1}}{n+p}\right|$

$=\left|\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}\ldots\ldots\ldots.+\frac{(-1)^{n-1}}{n+p}\right|$ now

$\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}\ldots\ldots\ldots.+\frac{(-1)^{n-1}}{n+p}=\frac{1}{(n+1)(n+2)}-\frac{1}{(n+3)(n+4)}+\cdots$(the last term will positive whether p is odd or even) (i.e) >0.

Hence $\left|S_{n+p}-S_n\right|=\frac{1}{(n+1)}-\frac{1}{(n+2)(n+3)}-\frac{1}{(n+3)(n+4)}$....(the last term will be positive whether p is even or odd) $<\frac{1}{(n+1)}<\varepsilon$ provided n > ($\frac{1}{\varepsilon}$ -1)

Let m be any integer >($\frac{1}{\varepsilon}$ -1), then we have $\left|S_{n+p}-S_n\right|<\varepsilon\forall n\geq m\ and$ p>0

1.7.2 Theorem:

Every Cauchy Sequence is bounded.

Proof: Let $\{S_n\}$ be Cauchy sequence for given $\varepsilon=I$ $\exists$ such that

$I^+ \ni |S_n - S_m| < q \; \forall n \geq m$

Implies $S_{m-1} < S_n < S_{m+1} \forall n \geq m$. Let k= Min $\{S_1 . S_2 . S_3 \ldots \ldots S_{m-1}\}$,

K= Max $\{S_1 . S_2 . S_3 \ldots \ldots S_{m-1}\}$ Then $k \leq S_n \leq K \forall n \geq N$. Hence S_n is bounded.

1.8 Series:

A series is a sum of elements. The sum can be finite or it can be infinite. The elements of the series can be either numbers or letters or a combination of both. A series can be represented

(a) By listing several elements along with the appropriate sign (+ or -) between the elements (or)

(b) By using what is called sigma notation with only the general term and the range of summation indicated.

Examples: 1.8.1

(1) 1 +2 +3………. +n

(2) $\sum_{n=1}^{10} (-1)^{n+1} n$. Both examples represent the same series.

As with sequences the main areas of interest with series are:

(a) The determination of the general term of the series if the general term is not given, and

(b) Finding out whether the sum of the given series exists.

1.8.2 Series Tests:

The Series tests are as follows:

General (n^{th}) Term Test (also known as the Divergence Test):

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Note: This test is a test for divergence only, and says nothing about convergence.

1.8.3 Geometric Series Test:

A geometric series has the form $\sum_{n=0}^{\infty} a r^n$, where "*a*" is some fixed scalar (real number). A series of this type will converge if $|r| < 1$, and the sum is $\frac{a}{1-r}$. A proof of this result follows. Consider the k^{th} partial sum and "*r*" times the k^{th} partial sum of the series

$$\begin{aligned} S_k &= a + ar^1 + ar^2 + ar^3 + \cdots + ar^k \\ r\,S_k &= ar^1 + ar^2 + ar^3 + \cdots + ar^k + ar^{k+1} \end{aligned}$$

The difference between rS_k and S_k is $(r-1)S_k = a\left(r^{k+1} - 1\right)$.

If r≠ 1, we can divide by (r - 1), to obtain $S_k = \frac{a\left(r^{k+1}-1\right)}{(r-1)}$.

Since the only place that "k" appears on the right in this last equation is in the numerator, the limit of the sequence of partial sums $\{S_k\}$ will exist if the limit as $S_k \to \infty$ exists as a finite number. This is possible if $|r| < 1$, and if this is true then the limit value of the sequence of partial sums, and hence the sum of the series, is $S = \frac{a}{1-r}$.

Sample Problem:1

Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$.

The general term a_n can be rewritten as $a_n = \frac{1}{n^2+n} = \frac{1}{n} - \frac{1}{n+1}$. We now consider the partial sums S_1, S_2, S_3, ... S_n, ... until a pattern emerges and then the limit value S will be determined.

$$
\begin{array}{lclcl}
S_1 & = & 1-\frac{1}{2} & = & 1-\frac{1}{2} \\
S_2 & = & \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right) & = & 1-\frac{1}{3} \\
S_3 & = & \left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right) & = & 1-\frac{1}{4} \\
S_4 & = & \left(1-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right) & = & 1-\frac{1}{5} \\
\vdots & & \vdots & & \vdots \\
S_n & = & \left(1-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right) & = & 1-\frac{1}{n+1}
\end{array}
$$

Since we have now determined the general pattern, the limit value S of the sequence of partial sums, and hence the sum of the series is seen to have a value of "1".

<u>1.8.4 Integral Test:</u>

Given a series of the form $\sum_{n=k}^{\infty} a_n$, set an = $f(n)$ where f(x) is a continuous function with positive values that are decreasing for x≥k. If the improper integral $\lim_{L\to\infty} \int_{x=k}^{L} f(x)\,dx$ exists as a finite real number, then the given series converges. If the improper integral above does not have a finite value, then the series above diverges. If the improper integral exists, then the following inequality is always true

$$\int_{x=p+1}^{\infty} f(x)\,dx \le \sum_{n=p}^{\infty} a_n \le a_p + \int_{x=p}^{\infty} f(x)\,dx$$

By adding the terms from n = k to n = p to each expression in the inequalities above it is possible to put both upper and lower bounds on the sum of the series. Also, it is possible to estimate the error generated in estimating the sum of the series by using only the first "p" terms. If the error is represented byR_p, then it follows that $\int_{x=p+1}^{\infty} f(x)\,dx \leq R_p \leq \int_{x=p}^{\infty} f(x)\,dx$.

1.8.5 Convergent and divergent series:

An infinite series $\sum_{n=1}^{\infty} U_n$ this said to be convergent if associate sequence of n is the partial sum is convergent and it is denoted by $\sum_{n=1}^{\infty} U_n = S_n$ where is S_n sum of the series. An infinite series $\sum_{n=1}^{\infty} U_n$ is said to be divergent to $+\infty$ (or)$-\infty$ according as S_n diverges to $+\infty$ (or)$-\infty$ respectively. An infinite series $\sum_{n=1}^{\infty} U_n$ is said to be oscillates finitely (or) infinitely according as a S_n oscillates finitely (or) infinitely.

Problem1:

Discuss the convergence of series 1+2+3+4+……. +n+……..∞

Solution: Let $\{S_n\}$ be the partial sum of n terms of the given series S_n =1+2+3+4+……..+n

$S_n=\frac{n(n+1)}{2}$, $\lim_{n\to\infty} S_n=\lim_{n\to\infty} \frac{n(n+1)}{2}$

$\lim_{n\to\infty} S_n=\infty$ therefore S_n is divergent

Problem2:

Discuss the nature of series 2-2+2-2+2……….

Solution: Let S_n be the partial sum of n terms of the given series. S_1=2,S_2=2-2=0

S_3=2-2+2=2, $S_n = \begin{cases} 0, & if\ n\ is\ even \\ 2, & if\ n\ is\ odd \end{cases}$,$\{S_n\}$ does not tend to unique limit. Therefore $S_n$is oscillatory finite. Hence the given series neither convergent nor divergent.

1.9 Comparison Tests:

There are four comparison tests that are used to test series. There are two convergence tests, and two divergence tests. To use these tests, it is necessary to know a number of convergent series and a number of divergent series. For the tests that follow, we shall assume that $\sum_{n=1}^{\infty} c_n$ is some known convergent series, that $\sum_{n=1}^{\infty} d_n$ is some known divergent series, and that $\sum_{n=1}^{\infty} a_n$ is the series to be tested. Also, it is to be assumed that for n$\in$ {1, 2, 3... (k-1)} the values are finite, and that each of the series contains only positive terms.

1.9.1 Standard Comparison Tests:

Convergence Test: If $\sum_{n=1}^{\infty} c_n$ is a convergent series and $a_n \le c_n$ for all n$\ge k$, then $\sum_{n=1}^{\infty} a_n$ is a convergent series.

Divergence Test: If $\sum_{n=1}^{\infty} d_n$ is a divergent series and $a_n \ge d_n$ for all n$\ge k$, then $\sum_{n=1}^{\infty} a_n$ is a divergent series.

1.9.2 Limit Comparison Tests:

Convergence Test: If $\sum_{n=1}^{\infty} c_n$ is a convergent series and $\lim_{n\to\infty} \frac{a_n}{c_n} = L$ where 0 $\le L < \infty$, then $\sum_{n=1}^{\infty} a_n$ is a convergent series.

Divergence Test: If $\sum_{n=1}^{\infty} d_n$ is a divergent series and $\lim_{n\to\infty} \frac{a_n}{d_n} = L$ Where 0 $< L \le \infty$, then $\sum_{n=1}^{\infty} a_n$ is a divergent series. The choice for the reference series $\sum_{n=1}^{\infty} c_n$ or $\sum_{n=1}^{\infty} d_n$ is often the geometric series $\sum_{n=0}^{\infty} ar^n$ or the hyper harmonic series (or p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges absolutely when p>1 and diverges otherwise. A special case is the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, which diverges (p = 1).

[The alternating p-series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges absolutely when p>1,

Converges conditionally when 0<p≤1 and diverges otherwise]

1.9.3 Alternating Series Test:

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ... + a_{(k-1)} + \sum_{n=k}^{\infty} a_n$ where a_1 , a_2 , a_3 , ... , $a_{(k-1)}$ can be any finite real numbers, and $\frac{a_{n+1}}{a_n} < 0$ for all $n \ge k$,if $\lim_{n\to\infty} a_n = 0$, then the series converges. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.

1.9.4 Ratio Test:

Given a series $\sum_{n=1}^{\infty} a_n$ with no restriction on the values of the a_n's except that they are finite, and that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, the series converges absolutely whenever $0\le L< 1$, diverges whenever $1 < L \le \infty$, and the test fails if L = 1.

1.9.5 Root Test:

Given a series $\sum_{n=1}^{\infty} a_n$ with no restriction on the values of the $a_n's$ except that they are finite, and that $\lim_{n\to\infty}\left(|a_n|\right)^{\frac{1}{n}} = L$, the series converges whenever $0\le L< 1$, diverges whenever $1 <L\le\infty$, and the test fails if L = 1.

1.9.6 Comparison test:

The theorems we state and prove below, enable us to analyse the behaviour or convergence or divergence of a given term series, by comparison with some suitable positive term series, whose behaviour is already known to us. For this reason, the tests that the theorems provide, are called comparison tests.

First Comparison test: Let $\sum u_n$ and $\sum v_n$ be two positive term series, such that $\sum v_n$ is convergent, and $u_n \le$ A v_n, for all $n\ge m$. A being a positive constant then $\sum u_n$ is convergent.

Proof: for each positive integer n, let

$U_n=u_1 + u_2 \ldots\ldots +u_n$ ………………………..……….(1)

$V_n=v_1 + v_2 \ldots\ldots +v_n$ …………………………..……..(2)

So, that, $< U_n>$ and, $< V_n>$ are sequence of partial sums of the two positive term series $\sum u_n$ and $\sum v_n$ respectively. As each, $< U_n>$ and, $< V_n>$ are both monotone increasing sequence

Now, we know A>0, such that $u_n \le A v_n$

for all $n\ge m$ ……………………….(3) $putting\; n =$ m+1, m+1…….(n-1), n and adding the numbers on respective side.

$u_{m+1} + u_{m+2} \ldots\ldots +u_n \le$ A $v_{m+1} + v_{m+2} \ldots\ldots +v_n$ (or)

$U_n - U_m \le$ A $(V_n - V_m)$ or $U_n \le$ A $V_n + U_m -$ A V_m $n\in N$ …. (4)

Since $\sum v_n$ is convergent, the sequence, $< V_n >$ of partition sums is bounded above, so that for some V>0, $V_n \leq V \ \forall$ n$\in N$ ……………………. (5)

From 4 and 5 $\quad U_n \leq$ A V +$U_m -$ A $V_m \forall$ n$\in N$. . ………. . ……. (6)

or, $< U_n >$ is bounded above. Hence $\sum u_n$ *is* convergent.

1.9.7 Theorem:

Let $\sum u_n$ and $\sum v_n$ be two positive term series, such that

i) $\sum v_n$ is divergent, and

ii) $u_n \geq$Bv_n, for all n$\geq m$. B being a positive constant then $\sum u_n$ *is* divergent.

Proof: for each positive integer n, let

U_n=$u_1 + u_2 \ldots . . + u_n$ ………………..……….(1)

V_n=$v_1 + v_2 \ldots . . + v_n$ ………..…………………....(2)

So that, $< U_n >$and, $< V_n >$ are sequence of partial sums of the two positive term series $\sum u_n$ and $\sum v_n$ respectively. As such, $< U_n >$and, $< V_n >$ are both monotone increasing sequence

Now, we know A >0, such that $u_n \geq$ B$v_n \forall$ n$\geq m$ ……………………. ……. . .(3)

putting $n =$m+1, m+1…….(n-1),n and adding the numbers on respective sides

$u_{m+1} + u_{m+2} \ldots . . + u_n \geq$ B $(v_{m+1} + v_{m+2} \ldots . . + v_n)$

or$U_n - U_m \geq$ B $(V_n - V_m)$ or $U_n \geq$ B V_n +$(U_m -$ B$V_m)$ $\forall$ n$\in N$ ……….(4)

Since $\sum v_n$ is divergent, the sequence, $< V_n >$ is not bounded above, ∴ for each G>0, however large there exists n$\in N$, *such that* $V_n > G$ ……. (5). From 4 and 5 large $\exists$ n$\in N$, *such that* $U_n \geq$ B G +$V_m -$ ABV_m. Since G>0 is arbitrary, $< U_n >$ is not bounded above. Hence $\sum u_n$ *is* divergent.

1.9.8 Theorem:

Let $\sum u_n$ and $\sum v_n$ be two positive term series, such that for some positive constant A and B, B$\leq \frac{u_n}{v_n} \leq A$ for all n $\geq m$ then the two series converge or diverge together.

Proof: We have 0 < B$\leq \frac{u_n}{v_n} \leq A \forall$ n$\geq m$

Since $v_n > 0 \forall$ n

∴ 0 < B$v_n \leq u_n \leq Av_n \forall$1 n$\geq m$ …………………….. (1)

Case 1: Let $\sum v_n$ be convergent

Also from (1) $u_n \leq Av_n \forall$ n$\geq m$

∴ By first comparison test for convergence

$\sum u_n$ Be also convergent …………………..…………..(2)

Case 2: Let $\sum v_n$ be divergent

Also from (1) $u_n \geq Bv_n \forall$ n$\geq m$

∴ By first comparison test for divergence

$\sum u_n$ Be also divergent ……………………….…………….(3)

Thus $\sum u_n$ is convergent if $\sum v_n$ is convergent and $\sum u_n$ is divergent

if $\sum v_n$ is divergence …………………………………… (4)

Now the inequalities (1) can also be put in the form

$0 < \frac{1}{A} \leq \frac{v_n}{u_n} \leq \frac{1}{B} \forall$ n$\geq m$…..………………..……....……..(5)

∴ The role of $\sum u_n$ and $\sum v_n$ in (4)$\sum v_n$ is convergent if $\sum u_n$ is convergent and $\sum v_n$ is divergent

if $\sum u_n$ is divergent …………………………………… (6)

From (4) and (6) the two series $\sum u_n$ *and* $\sum v_n$ converge or diverge together.

1.9.9 Theorem:

Let $\sum u_n$ and $\sum v_n$ be two positive term series if $\lim_{n\to\infty} \frac{v_n}{u_n}$ =l$\neq$ 0 then the two series converge or diverge together

Proof: since $u_n > 0$ *and* v_n>0 for all n$\in N$, ∴ $\frac{u_n}{v_n}$> 0 $\forall$ n$\in N$,

∴ if $\lim_{n\to\infty} \frac{v_n}{u_n}$ =l $\geq$ 0 …………………………….………(1)

But, it is known, then l $\neq$ 0, therefore l>0 …………… (2)

Now, let $\varepsilon > 0$ be chosen, such that l- ε>0. Then there exits m$\in N$, such that

$$\left|\frac{u_n}{v_n} - 1\right| < \varepsilon \forall n \geq m$$

(or)

l- $\varepsilon < \frac{u_n}{v_n} <$ l+ $\varepsilon \forall$ n$\geq m$ ………………………………(3)

Putting B= l- ε, A= l+ ε , we have positive constants A and B, such that

B <$\frac{u_n}{v_n}$<A $\forall$ n$\geq m$ ……………………………………. (4)

Since v_n>0 for all n,v_n B <,u_n< A v_n for all n$\geq m$. Hence the series $\sum u_n$ and $\sum v_n$ converge or diverge together

1.9.10 Second Comparison test:

Theorem: Let $\sum u_n$ and $\sum v_n$ be two positive term series, such that

$i)$ $\sum v_n$is convergent, and

ii) $\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}$, for all sufficiently large values of n. then $\sum u_n$ is convergent.

Proof: Let m ϵ N, such that$\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}$,$\forall$ n$\geq m$

Putting n=m+1, m+2, m+3,………..(n-1), we get

$\frac{u_{m+1}}{u_m} \leq \frac{v_{m+1}}{v_m}$,$\frac{u_{m+2}}{u_{m+1}} \leq \frac{v_{m+2}}{v_{m+1}}$ …………………$\frac{u_n}{u_{n-1}} \leq \frac{v_n}{v_{n-1}}$, multiplying the numbers on the respective sides

$\frac{u_n}{u_m} \leq \frac{v_n}{v_m}$ $\forall$ n$\geq m$ (or) $u_n \leq (\frac{u_m}{v_m})\, v_n$ $\forall$ n$\geq m$

Since $\sum v_n$ is convergent and $(\frac{u_m}{v_m})$ is a positive constant. ∴ By first comparison test, $\sum u_n$ is also convergent.

1.9.11 Theorem:

Let $\sum u_n$ and $\sum v_n$ be two positive term series, such that

$i)$ $\sum v_n$is divergent, and

ii) $\frac{u_{n+1}}{u_n} \geq \frac{v_{n+1}}{v_n}$, for all sufficiently large values of n. then $\sum u_n$ is divergent.

Proof: Let m ϵ N, such that $\frac{u_{n+1}}{u_n} \geq \frac{v_{n+1}}{v_n}$ $\forall$ n$\geq m$

Putting n=m+1, m+2, m+3,………..(n-1), we get

$\frac{u_{m+1}}{u_m} \geq \frac{v_{m+1}}{v_m}$,$\frac{u_{m+2}}{u_{m+1}} \geq \frac{v_{m+2}}{v_{m+1}}$ …………………$\frac{u_n}{u_{n-1}} \geq \frac{v_n}{v_{n-1}}$, multiplying the numbers on the respective sides$\frac{u_n}{u_m} \geq \frac{v_n}{v_m}$ $\forall$ n$\geq m$ (or) $u_n \geq (\frac{u_m}{v_m})\, v_n$ $\forall$ n$\geq m$.

Since $\sum v_n$ is divergent and $(\frac{u_m}{v_m})$ is a positive constant. By first comparison test, $\sum u_n$ is also divergent.

Example: 1

Examine for convergence the series $\frac{1}{3.7}+\frac{1}{4.9}+\frac{1}{5.11}+\cdots$..

Solution: The nth term $u_n$of the series is

$u_n=\frac{1}{(n+2)(2n+5)}$>0 $\forall$ n$\in N$. Therefore, the given series $\sum u_n$ *is* of positive terms

Now $u_n=\left[\frac{1}{n^2}\right]\left[\frac{1}{\left(1+\frac{2}{n}\right)\left(2+\frac{5}{n}\right)}\right]$ $=\left[\frac{1}{2n^2}\right]\left[\left(1+\frac{2}{n}\right)\right]^{-1}\left[\left(1+\frac{5}{2n}\right)\right]^{-1}$

$=\left[\frac{1}{2n^2}\right]\left(1-\frac{2}{n}+\frac{4}{n^2}\ldots\right)\left(1-\frac{5}{2n}-\cdots\right)$ for large value of n, $\frac{1}{n^2}$ is small and u_n behaves like $\frac{1}{n^2}$

Let $\sum v_n$ be the series, where $v_n=\frac{1}{n^2}$. Then we know $\sum v_n$ is convergent.

Now $\frac{u_n}{v_n}=\left[\frac{1}{2n^2}\right]\left(1-\frac{2}{n}+\frac{4}{n^2}-\cdots\right)\left(1-\frac{5}{2n}-\cdots\right)n^2$

$\therefore \lim_{n\to\infty}\frac{u_n}{v_n}=\frac{1}{2}$(1-0) (1-0) $=\frac{1}{2}(\neq 0$. $\therefore$ by first comparison test, the two series converge or diverge together. Since $\sum v_n$ is convergent. $\therefore$ The given series $\sum u_n$ *is* also convergent.

Example: 2 Show that the series $\frac{1}{5}+\frac{\sqrt{2}}{7}+\frac{\sqrt{3}}{9}+\frac{\sqrt{4}}{11}+\cdots$.. is divergent

Solution: Here, the nth term $u_n$of the series is $u_n=\frac{\sqrt{n}}{(2n+3)}$>0, $\forall$ n$\in N$. Therefore $\sum u_n$ *is* of positive term series

Now, $u_n=\frac{\sqrt{n}}{(2n)}\left[\left(1+\frac{3}{2n}\right)\right]^{-1}=\frac{1}{(2\sqrt{n})}\left(1-\frac{3}{2n}+\frac{9}{4n^2}-\cdots\right)$

for large value of n, u_n behaves as $\frac{1}{\sqrt{n}}$. Let $v_n=\frac{1}{\sqrt{n}}$.and consider the series $\sum v_n$ which is of type $\sum\frac{1}{n^p}$ with p=$\frac{1}{2}$<1 and hence divergent

$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\sqrt{n}\frac{1}{(2\sqrt{n})}\left(1-\frac{3}{2n}+\frac{9}{4n^2}-\cdots\right)=\frac{1}{2}(\neq 0)$

Therefore, by first comparison test, the two series$\sum u_n$ *and* $\sum v_n$is converge or diverge together.

Since $\sum v_n$ is divergent, Therefore the given series $\sum u_n$ *is* also divergent.

1.9.12 D'ALEMBERT'S RATIO TEST:

This test is due to the French mathematician Jean Le Rand D'ALEMBERT'S

Theorem: Let $\sum u_n$ be a positive term series and suppose $\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}}$ =l then

1. If l >1, the series is convergent, and
2. If l <1, the series is divergent,
3. If l =1, the test fails.

Proof: We know u_n> 0 $\forall$n$\in N$

$\therefore \frac{u_n}{u_{n+1}}$>0 , $\forall$ n$\in N$

$\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}} = l$ then $l \geq 0$

Case 1: Suppose l >1. We choose $\varepsilon > 0$ be such that l - ε >1.

Now $\lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}} = l$ therefore given $\varepsilon > 0, \exists$ m$\in N$, such that

$\left|\left(\frac{u_n}{u_{n+1}} - l\right)\right| < \varepsilon \forall n \geq m$

(or) - $\varepsilon < \frac{u_n}{u_{n+1}}$ <l+$\varepsilon \forall$ n$\geq m$

Setting - ε =r we have r> 1 or $\frac{1}{r}$<1

Now, we have

$\frac{u_n}{u_{n+1}} > r \ \forall$ n$\geq m$

$(or)\ \frac{u_n}{u_{n+1}} < \frac{1}{r}\frac{1}{r}$<1 $\forall$ n$\geq m$.

$putting\ n$ =m+1, m+1…….(n-1), we get $\frac{u_{m+1}}{u_n} < \frac{1}{r}\frac{u_{m+2}}{u_{m+1}} < \frac{1}{r}$

……………………..

……………………….

$$\frac{u_n}{u_{n-1}} < \frac{1}{r}$$

Multiplying the numbers on the respective sides $\frac{u_n}{u_m} < \left(\frac{1}{r}\right)^{n-m}$ (or)

$u_n < u_m \left(\frac{1}{r}\right)^{-m} u_m \left(\frac{1}{r}\right)^n$ .Now the geometric series $\sum \left(\frac{1}{r}\right)^n$ with common ratio $\left(\frac{1}{r}\right)$<1 is convergent

Therefore, by first comparison Test

$\sum u_n$ is convergent, if l>1

Case 2: Let l <1, we can choose $\varepsilon > 0$ such that l+ ε<

$Now \lim_{n\to\infty} \frac{u_n}{u_{n+1}}$ =l ∴given $\varepsilon > 0$∃ m∈ N, such that, $\left|\left(\frac{u_n}{u_{n+1}} - l\right)\right|$<$\varepsilon$∀ n≥ m

(or) l- ε<$\frac{u_n}{u_{n+1}}$ <l+ε∀ n≥ m

Setting l+ ε =ρ we have ρ< 1 or $\frac{1}{\rho}$>1

Now, we have$\frac{u_n}{u_{n+1}} < \rho$ for all n≥ m (or) $\frac{u_n}{u_{n+1}} > \frac{1}{\rho}$for all n≥ m.$\frac{1}{\rho}$<1

$putting\ n =$m+1, m+1……. (n-1), and multiplying the sides of resulting inequalities, we get

$\frac{u_n}{u_m} > \left(\frac{1}{\rho}\right)^{n-m}$ n >m $(or) u_n > u_m \rho^{-m} \left(\frac{1}{\rho^n}\right)^1$

Since $\left(\frac{1}{\rho}\right)$.1, ∴ $\left(\frac{1}{\rho^n}\right) \to \infty$ as n→ ∞

Thus $\lim_{n\to\infty} u_n$ ≠0

Hence, $\sum u_n$ is divergent, if l<1

***Case* 3:** Let l=1. Consider the two series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$

If u_n =$\frac{1}{n}$ then $\frac{u_n}{u_{n+1}}$ =$\frac{n+1}{n} \to 1$ as n→ ∞

f u_n =$\frac{1}{n^2}$ then $(\frac{n+1}{n})^2 \to 1$ as n→ ∞

Thus $\frac{u_n}{u_{n+1}} \to 1$ in each case, but $\sum \frac{1}{n}$ is divergent while $\sum \frac{1}{n^2}$convergent. Therefore, when l=1. The test fails

Example: 1. Examine the convergent the series$\sum u_n$, where $u_n = \frac{10^n}{n}$

Solution: The nth term $u_n = \frac{10^n}{n}$> 0 for all n ∈ N

Therefore:$\sum u_n$, is a positive term series. Now $u_n = \frac{10^n}{n}, u_{n+1} = \frac{10^{n+1}}{n+1}$

Therefore$\left[\frac{u_n}{u_{n+1}}\right]$=$\frac{10^n}{n} \frac{n+1}{10^{n+1}}$ =$\left[1 + \frac{1}{n}\right]\left(\frac{1}{10}\right)$

$\lim_{n\to\infty}\left[\frac{u_n}{u_{n+1}}\right]=\lim_{n\to\infty}\left[1+\frac{1}{n}\right]\left(\frac{1}{10}\right)=(1+0)\left(\frac{1}{10}\right)=\left(\frac{1}{10}\right)<1$

Therefore, By D'alembert's Ratio Test, the given series $\sum u_n$ is divergent

Example: 2 Examine the convergence the series

$1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\ldots\ldots$

Solution: The nth term $u_n=\frac{n^p}{n!}>0$ for all n$\in N$

Therefore$\sum u_n$, is a positive term series. Now $u_n=\frac{n^p}{n!}, u_{n+1}=\frac{(n+1)^p}{(n+1)!}$

Therefore$\left[\frac{u_n}{u_{n+1}}\right]=\frac{n^p}{n!}\frac{(n+1)!}{(n+1)^{p+1}} \quad = \frac{n+1}{(1+\frac{1}{n})^p}$

As $n\to\infty$, $(1+\frac{1}{n})^p$ →1 and ($n+1)\to\infty$

$\lim_{n\to\infty}\left[\frac{u_n}{u_{n+1}}\right]$=+∞>1

∴The given series $\sum\frac{n^p}{n!}$ is convergent

Example: 3 Examine the convergence the series

$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots+\frac{x^n}{n!}+\ldots\ldots\ldots$ x>0

Solution: The nth term $u_n=\frac{x^n}{n!}>0$ for all n$\in N$ since x>0

∴ $\sum u_n$, is a positive term series. Now $u_n=\frac{x^n}{n!}, u_{n+1}=\frac{(x)^{n+1}}{(n+1)!}$

∴ $\left[\frac{u_n}{u_{n+1}}\right]=\frac{x^n}{n!}\frac{(n+1)!}{(x)^{n+1}} \quad = \frac{n+1}{x}$

Since $n+1\to\infty, as\ n\to\infty$

$\lim_{n\to\infty}\left[\frac{u_n}{u_{n+1}}\right]$=+∞>1 for all x > 0

∴ By D'alembert's Ratio Test, the given series $\sum u_n$ is convergent for all x > 0

1.9.13 Cauchy's root test

We shall now introduce several intrinsic tests of convergence. Each of them dependents on the items of the given itself. We begin with Cauchy's root test

Theorem:

Let $\sum u_n$ be a positive term series and suppose $\lim_{n\to\infty}\sqrt[n]{u_n}$ =l then

1. If l<1, the series is convergent, and
2. If l>1, the series is divergent,
3. If l=1, the test fails.

Proof:

Case 1: Suppose l<1. Let $\varepsilon > 0$ be such that l< l+ ε <1. Setting l+ $\varepsilon = r$, we have r <1

Now $\lim_{n\to\infty} u^{\frac{1}{n}}$ =l,∴ $\varepsilon > 0, \exists$ m∈ N, such that

$\left|u_n^{\frac{1}{n}} - l\right| < \varepsilon \forall$ n≥ m (or) l- $\varepsilon < u^{\frac{1}{n}} <$ l+ $\varepsilon \forall$ n≥ m, $in\ particular$

$u_n^{\frac{1}{n}} < r =$ l+ $\varepsilon \forall$ n≥ m

(or)$u_n < r^n \forall$ n≥ m

Since r<1, the geometric series Let $\sum r^n$ converges

∴ By comparison test $\sum u_n$ is convergent

Case 2: Suppose l >1. Let $\varepsilon > 0$ be such that l - ε >1. Setting l- $\varepsilon = \rho$, we have ρ>1.

Now $\lim_{n\to\infty} u^{\frac{1}{n}}$ =l, ∴ given $\varepsilon > 0, \exists$ m∈ N, such that, $\left|u_n^{\frac{1}{n}} - l\right| < \varepsilon \ \forall$ n≥ m

(or) l- $\varepsilon < u^{\frac{1}{n}} <$ l+ $\varepsilon \forall$ n≥ m

$in\ particular u_n^{\frac{1}{n}} > \rho =$ l- $\varepsilon \forall$ n≥ m (or) $u_n < \rho^n \forall$ n≥ m

Since ρ> 1, the geometric series Let $\sum \rho^n$ diverges

By comparison test $\sum u_n$ is divergent

Case 3: Suppose l=1.We shall now that the test limits, which means the test fails to give a clear conclusion. We consider two examples

A) $\sum \frac{1}{n}$, here $u_n = \frac{1}{n}$ and $(u_n)^{\frac{1}{n}} = \frac{1}{u^{\frac{1}{n}}}$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} = \frac{1}{u^{\frac{1}{n}}}$$

B) $\sum \frac{1}{n^2}$, here $u_n = \frac{1}{r}$ and $(u_n)^{\frac{1}{n}} = \frac{1}{(n^2)^{\frac{1}{n}}}$

$$\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} = \frac{1}{u^{\frac{2}{n}}} = \frac{1}{\lim_{n\to\infty} n^{\frac{2}{n}}} = \frac{1}{1} = 1$$

Thus, we each value $\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = 1$,but, we know otherwise, $\sum\frac{1}{n}$ is divergent write $\sum\frac{1}{n^2}$ is convergent.Hence, the test fails when l=1

Example 1: Examine for convergent the series

1+$\frac{1}{2^2}$ +$\frac{1}{3^3}$ +$\frac{1}{4^3}$ +…………..

Solution:

Here, the n^{th} term u_n of the series is

$u_n = \frac{1}{n^n}$>0 ∀l n∈ N

$\sum u_n$ is a positive term series?

$(u_n)^{\frac{1}{n}}$ =$(\frac{1}{n^n})^{\frac{1}{n}}$ =$\frac{1}{n}$ and $\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = 0 < 1$

Therefore, Cauchy's root test, the series $\sum u_n$ is convergent

Example 2: Examine for convergent the series whose nth term is

$$((\frac{n+1}{n})^{n+1} - (\frac{n+1}{n})^1)^{-n}$$

Solution: The nth term u_n of the series is $u_n = ((\frac{n+1}{n})^{n+1} - (\frac{n+1}{n})^1)^{-n}$

$$=((\frac{n+1}{n})^{-n}\left([\frac{n+1}{n}]^n - 1\right)^{-n}$$

$$(u_n)^{\frac{1}{n}} = \left(1+\frac{1}{n}\right)^{-1}[(1+\frac{1}{n})^n - 1)]^{-1}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-1}\lim_{n\to\infty}[(1+\frac{1}{n})^n - 1)]^{-1}$$

$$=1\ \frac{1}{e-1}$$

$$=\frac{1}{e-1}<1$$

∴ By Cauchy's root test, the series $\sum u_n$ is convergent

Example 3 Examine for convergent, the series

$\frac{1}{3} + \frac{x}{36} + \frac{x^2}{243} +$ …………+$\frac{x^{n-1}}{(3^n n^2)}$+…….. x> 0

Solution: Here, the nth term u_n of the series is u_n=$\frac{x^{n-1}}{(3^n n^2)}$>0 for all n∈ N,

$u_n^{\frac{1}{n}} = \frac{x^{1-\frac{1}{n}}}{\left(3\ n^{\frac{2}{n}}\right)}$,

$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{[x^{1-\frac{1}{n}}]}{[\left(3\ n^{\frac{2}{n}}]\right)}$

$= \frac{x}{3.1} = \frac{x}{3}$ By Cauchy's root test, the series is convergent if x<3 and diverges x>3.Now, when x = 3, $u_n = \frac{x^{n-1}}{(3^n n^2)}$

$$= \frac{1}{3} \frac{1}{n^2}$$

Let $v_n = \frac{1}{n^2}$ then $\sum v_n$ is of the type $\sum \frac{1}{n^2}$ with p =2 and therefore convergent

∴, by comparison test, $\sum u_n$ is convergent

Hence, the given series is

i) Convergent, if x $\leq$ 3

ii) Divergent if x $>$ 3

1.9.14 Conditional and Absolute Convergence:

A convergent series that contains an infinite number of both negative and positive terms must be tested for absolute convergence. A series of the form $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ the series of absolute values is convergent. If $\sum_{n=1}^{\infty} a_n$ is convergent, but $\sum_{n=1}^{\infty} |a_n|$ the series of absolute values is divergent, then the series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

Note:

In some cases, it is easier to show that $\sum_{n=1}^{\infty} |a_n|$ is convergent. It then follows immediately that the original series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. This series converges absolutely when,

$$0 \leq |x+3| < \frac{1}{\lim_{n\to\infty} \left|\frac{n+5}{n+4}\right|}$$

$$\text{i.e.} \quad 0 \leq |x+3| < 1$$

The radius of convergence is $R = 1$. When $(x + 3) = 1$, the given series becomes $\sum_{n=0}^{\infty} \frac{1}{n+4}$ which is a divergent series. When $(x + 3) = -1$, the given series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+4}$ which is a [conditionally] convergent alternating series. Hence, the series will converge whenever $-1 \leq x+3 < 1$. This can also be expressed by saying that the interval of convergence I for this series is $I = \{x \mid -4 \leq x < -2\}$, or $I = [-4, -2)$.

1.9.15 Conditional and absolute convergence:

Suppose $\sum u_n$ is a series of numbers of positive and negative signs (that is, arbitrary signs). By taking Absolute value $|u_n|$ of each term of the given series, we obtain a new series $\sum|u_n|$. Of course, $\sum|u_n|$ is a series of positive terms

Definition: 1 A series $\sum u_n$ is said to be absolute convergence if $\sum|u_n|$ is convergent

Definition: 2 A series $\sum u_n$ is said to be conditionally convergence if $\sum|u_n|$ is convergent but $\sum|u_n|$ is divergent

Examples:

1. The series $\sum u_n$, where $u_n = (-1)^{n-1}\left(\frac{1}{n^2}\right)$ is absolute convergence, here $u_n = (-1)^{n-1}\left(\frac{1}{n^2}\right)$

$|u_n| = \left(\frac{1}{n^2}\right)$. Now $\sum\left(\frac{1}{n^2}\right)$ is of type $\sum\left(\frac{1}{n^p}\right)$ with p=2 >1, which is convergent. $\therefore \sum|u_n|$ is convergent. Hence $\sum u_n$ is absolute convergence.

2. The series $\sum u_n$, where $u_n = (-1)^{n-1}\left(\frac{1}{n}\right)$ is conditionally convergence, here $u_n = (-1)^{n}\left(\frac{1}{n}\right)$

$|u_n| = \left(\frac{1}{n}\right)$. Now $\sum\left(\frac{1}{n}\right)$ is of type $\sum\left(\frac{1}{n^p}\right)$ with p=1, which is divergent. But, it can be shown by Leibnitz theorem that $\therefore \sum|u_n|$ is convergent $\sum(-1)^{n-1}\left(\frac{1}{n}\right)$ is convergent. Hence the series $\sum(-1)^{n-1}\left(\frac{1}{n}\right)$ is conditionally convergence

1.9.16 Theorem: An absolute convergent series is also convergent.

Proof: *Let* $\sum u_n$ *be a series,* which is absolute convergent. Then by definition

$\sum|u_n|$ is convergent………(1)

$\therefore$ By Cauchy's general principle of convergence, given $\varepsilon>0$, there exists m$\in$N, such that $\left||u_{n+1}|+|u_{n+2}|+|u_{n+3}|+\cdots\ldots\ldots+|u_{n+p}|\right|<\varepsilon$

for all n$\geq m, p\geq 1$.........(2)

Now, let us examine for convergence the series$\sum u_n$, let $\varepsilon>0$, then by (2)

$|u_{n+1}+u_{n+2}+\cdots\ldots\ldots u_{n+p}|\leq|u_{n+1}|+|u_{n+2}|+|u_{n+3}|+\cdots\ldots\ldots+|u_{n+p}|<\varepsilon$ provided n$\geq m$, $p\geq 1$. Hence, given $\varepsilon>0$,$\exists$ m$\in$N, such that $|u_{n+1}+u_{n+2}+\cdots\ldots\ldots u_{n+p}|<\varepsilon$

for all n $\geq m, p\geq 1$

$\therefore$ By Cauchy's general principle of convergence $\sum u_n$ is convergent

Example: 1 Show that the series $x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots+\frac{x^n}{n!}$ Converges absolutely for all values of x.

Solution: The nth term u_n of the series is $u_n=\frac{x^n}{n!}$ x$\in$R. taking absolute values $|u_n|=\frac{|x|^n}{n!}$ $|u_{n+1}|=\frac{|x|^{n+1}}{(n+1)!}$ Now $\left(\frac{|u_n|}{|u_{n+1}|}\right)$

$$=\frac{|x|^n}{n!}\frac{(n+1)!}{|x|^{n+1}}=\frac{(n+1)}{|x|}$$

Since $(n+1)\to\infty$ as n $\to\infty$

$$\therefore \lim_{n\to\infty}\left(\frac{|u_n|}{|u_{n+1}|}\right)=\lim_{n\to\infty}\frac{(n+1)}{|x|}$$

=+∞>1, for all x$\in$R and hence by D'Alenbents' Ratio test

$\sum u_n$ is convergent. $\therefore$ The given series$\sum u_n$ is absolutely convergent

Example: 2 Show that the series$\sum\frac{(-1)^{n-1}}{\sqrt{n}}$ is conditionally convergent.

Solution: the n^{th} term t_n of the series is

$$t_n=\left(\frac{(-1)^n}{\sqrt{n}}\right)u_n$$

where $u_n=\frac{1}{\sqrt{n}}\forall n\in N$

$\therefore$ The series is of alternating type. Now,

$u_{n+1} - u_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}} < 0 \ \forall\, n \in N$

$\therefore u_{n+1} < u_n \forall\, n \in N$. And hence $< u_n >$ is monotone decreasing.

Also $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = (\lim\limits_{n\to\infty} \frac{1}{n})^{\frac{1}{2}}$ =0. $\therefore$ Leibnitz theorem, the given series $\sum(-1)^n u_n$ is convergent ……………(1)

Now, $|u_n| = \frac{1}{\sqrt{n}} = \frac{1}{n^{\frac{1}{2}}}$ $\therefore \sum|u_n| = \sum \frac{1}{n^{\frac{1}{2}}}$ is of the type $\sum \frac{1}{n^p}$ with p=$\frac{1}{2}$<1, $\therefore \sum|u_n|$ is divergent

……..(2)

From (1) and (2) $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$ is conditionally convergent.

Example: 3 Show that the series $x - \frac{x^2}{3} + \frac{x^4}{5} + \ldots\ldots + \frac{x^n}{n+1}$ Converges if and only if

-1 < x ≤ 1

Solution: Here the n^{th} term u_n of the series is $u_n = x^{2n-1} \frac{(-1)^{n-1}}{2n-1}$ and

$|u_n| = \frac{|x|^{2n-1}}{2n-1}$. Let us examine the convergence of $\sum|u_n|$ we have

$$|u_n| = \frac{|x|^{2n-1}}{2n-1},$$

$$|u_{n+1}| = \frac{|x|^{2n+1}}{2n+1} \therefore$$

$$\frac{|u_n|}{|u_{n+1}|} = \frac{|x|^{2n-1}}{2n-1} \frac{|x|^{2n+1}}{2n+1}$$

$$= \frac{2n+1}{2n-1} \frac{1}{|x|^2}$$

$$= \lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|}$$

$$= \lim_{n\to\infty} \frac{2n+1}{2n-1} \frac{1}{|x|^2} = \lim_{n\to\infty} \frac{1+\frac{1}{2n}}{1-\frac{1}{2n}} \frac{1}{|x|^2}$$

$$= \frac{1}{|x|^2}$$

$\therefore$ by D'alembert's test, the series $\sum u_n$ is convergent if $\frac{1}{|x|}$>1 (or) $|x| < 1$ and the series is$\sum u_n$ is divergent if $\frac{1}{|x|}$<1 (or) $|x| > 1, \therefore$ The series absolutely convergent if $|x| < 1,$ and hence, the series is convergent for all x, -1<x<1 when x = 1, the series becomes

1-$\frac{1}{3}+\frac{1}{5}$-……… This is an alternating series. The n^{th} term is $(-1)^{n-1}\frac{1}{2n-1}$ (or) is $(-1)^{n-1}u_n$, where $u_n = \frac{1}{2n-1}$>0

Now $u_{n+1}-u_n$=$\frac{1}{2n+1}-\frac{1}{2n-1}$<0 $\therefore$<u_n > is monotone decreasing

Also $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{1}{2n-1} = 0, \therefore by$ Leibnitz test, the series $\sum(-1)^{n-1}\frac{1}{2n-1}$is convergent. Again, when x=-1 the series becomes 1-$\frac{1}{3}+\frac{1}{5}$-………+$\frac{1}{2n-1}+\cdots$...... This series is known to be divergent. Hence the given series is convergent if and only if -1<x<1

UNIT - II

2.0 Differentiation:

Definition :

Let f be a function defined on an interval I: and c$\in I.Then$

$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$,c+h $\in I$ and h≠0. If it exists, is called the derivative of f at c, and denoted by f'(c) or D f (c). Also, f is said to be derivative at c, if f has a derivative at c

Notations for the Derivative and rules:

The derivative of $y = f(x)$ may be written in any of the following ways:

$$f'(x), \quad y', \quad \frac{dy}{dx}, \quad \frac{d}{dx}[f(x)], \quad \text{or} \quad D_x[f(x)].$$

I. Basic Differentiation Rules

A. Suppose *c* and *n* are constants, and f *and* g are differentiable functions.

(1) $f(x) = cg(x)$

$$f'(x) = \lim_{b\to x}\frac{f(b)-f(x)}{b-x} = \lim_{b\to x}\frac{cg(b)-cg(x)}{b-x} = c\lim_{b\to x}\frac{g(b)-g(x)}{b-x} = cg'(x)$$

(2) $f(x) = g(x) \pm k(x)$

$$f'(x) = \lim_{b\to x}\frac{f(b)-f(x)}{b-x} = \lim_{b\to x}\frac{[g(b)\pm k(b)]-[g(x)\pm k(x)]}{b-x}$$

$$= \lim_{b\to x}\frac{g(b)-g(x)}{b-x} \pm \lim_{b\to x}\frac{k(b)-k(x)}{b-x} = g'(x) \pm k'(x)$$

(3) $f(x) = g(x)k(x)$

$$f'(x) = \lim_{b\to x}\frac{f(b)-f(x)}{b-x} = \lim_{b\to x}\frac{g(b)k(b)-g(x)k(x)}{b-x}$$

$$= \lim_{b\to x}\frac{g(b)k(b)-g(b)k(x)+g(b)k(x)-g(x)k(x)}{b-x}$$

$$= \left[\lim_{b\to x} g(b)\right]\left[\lim_{b\to x}\frac{k(b)-k(x)}{b-x}\right] + \left[\lim_{b\to x} k(x)\right]\left[\lim_{b\to x}\frac{g(b)-g(x)}{b-x}\right]$$

$= \; g(x)k'(x) + k(x)g'(x)$ (Product Rule)

(4) $f(x) = \dfrac{g(x)}{k(x)} \Rightarrow f(x)k(x) = g(x) \Rightarrow g'(x) = f(x)k'(x) + k(x)f'(x) \Rightarrow$

$$f'(x)=\frac{g'(x)-f(x)k'(x)}{k(x)}=\frac{g'(x)-\left[\frac{g(x)}{k(x)}\right]k'(x)}{k(x)}=\frac{k(x)g'(x)-g(x)k'(x)}{[k(x)]^2}.$$

This derivative rule is called the Quotient Rule.

(5) $f(x)=c$

$$f'(x)=\lim_{b\to x}\frac{f(b)-f(x)}{b-x}=\lim_{b\to x}\frac{c-c}{b-x}=\lim_{b\to x}\frac{0}{b-x}=\lim_{b\to x}0=0$$

(6) $f(x)=x$

$$f'(x)=\lim_{b\to x}\frac{f(b)-f(x)}{b-x}=\lim_{b\to x}\frac{b-x}{b-x}=\lim_{b\to x}1=1$$

(7) $f(x)=x^n$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

$$=\lim_{h\to 0}\frac{\left[x^n+nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\ldots\right]-x^n}{h}=\lim_{h\to 0}\left[\frac{nx^{n-1}h+h^2\left(\frac{n(n-1)}{2}x^{n-2}+\ldots\right)}{h}\right]$$

$$=\lim_{h\to 0}\left[nx^{n-1}+h\left(\frac{n(n-1)}{2}x^{n-2}+\ldots\right)\right]=nx^{n-1}\quad\text{(Power Rule)}$$

Example 1: Suppose f and g are differentiable functions such that $f(1)=3$, $g(1)=7$, $f'(1)=-2$, and $g'(1)=4$. Find (i) $(f+g)'(1)$, (ii) $(g-f)'(1)$, (iii) $(fg)'(1)$, (iv) $\left(\frac{g}{f}\right)'(1)$, and $\left(\frac{f}{g}\right)'(1)$.

(i) $(f+g)'(1)=f'(1)+g'(1)=-2+4=2$

(ii) $(g-f)'(1)=g'(1)-f'(1)=4-(-2)=6$

(iii) $(fg)'(1)=f(1)g'(1)+g(1)f'(1)=3(4)+7(-2)=12+(-14)=-2$

(iv) $\left(\frac{g}{f}\right)'(1)=\frac{f(1)g'(1)-g(1)f'(1)}{[f(1)]^2}=\frac{3(4)-7(-2)}{3^2}=\frac{12+14}{9}=\frac{26}{9}$

(v) $\left(\frac{f}{g}\right)'(1)=\frac{g(1)f'(1)-f(1)g'(1)}{[g(1)]^2}=\frac{7(-2)-3(4)}{7^2}=\frac{-14-12}{49}=\frac{-26}{49}$

Example 2: If $f(x)=x^4-3x^3+5x^2-7x+11$, find $f'(x)$.

$$f'(x)=4x^3-3(3x^2)+5(2x)-7(1)+0=4x^3-9x^2+10x-7$$

Example 3: If $f(x)=4\sqrt{x}-\dfrac{3}{\sqrt[3]{x^2}}+\dfrac{5}{x}-\dfrac{7}{x^5}$, then find $f'(x)$.

$$f(x)=4\sqrt{x}-\frac{3}{\sqrt[3]{x^2}}+\frac{5}{x}-\frac{7}{x^5}=4x^{1/2}-3x^{-2/3}+5x^{-1}-7x^{-5}\Rightarrow$$

$$f'(x)=4\left(\tfrac{1}{2}x^{-1/2}\right)-3\left(-\tfrac{2}{3}x^{-5/3}\right)+5\left(-1x^{-2}\right)-7\left(-5x^{-6}\right)$$

$$=2x^{-1/2}+2x^{-5/3}-5x^{-2}+35x^{-6}=\frac{2}{\sqrt{x}}+\frac{2}{\sqrt[3]{x^5}}-\frac{5}{x^2}+\frac{35}{x^6}$$

Example 4: If $f(x)=\dfrac{x^2+2x-3}{3x-4}$, then find $f'(1)$.

$$f'(x)=\frac{(3x-4)(2x+2)-(x^2+2x-3)(3)}{(3x-4)^2}=\frac{6x^2-2x-8-3x^2-6x+9}{(3x-4)^2}$$

$$=\frac{3x^2-8x+1}{(3x-4)^2}\Rightarrow f'(1)=\frac{3(1)^2-8(1)+1}{[3(1)-4]^2}=\frac{-4}{1}=-4 \text{ (or)}$$

$$f'(1)=\frac{[3(1)-4][2(1)+2]-[1^2+2(1)-3](3)}{[3(1)-4]^2}=\frac{(-1)(4)-(0)(3)}{(-1)^2}=\frac{-4}{1}=-4$$

B. Trigonometric functions

(1) $f(x)=\sin x$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}=\lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}$$

$$=(\sin x)\left[\lim_{h\to 0}\frac{\cos h-1}{h}\right]+(\cos x)\left[\lim_{h\to 0}\frac{\sin h}{h}\right]=(\sin x)(0)+(\cos x)(1)=\cos x$$

(2) $f(x)=\cos x$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}=\lim_{h\to 0}\frac{\cos x(\cos h-1)-\sin x\sin h}{h}$$

$$=(\cos x)\left[\lim_{h\to 0}\frac{\cos h-1}{h}\right]-(\sin x)\left[\lim_{h\to 0}\frac{\sin h}{h}\right]=(\cos x)(0)-(\sin x)(1)$$

$$-\sin x$$

(3) $f(x)=\tan x=\dfrac{\sin x}{\cos x}$

$$f'(x)=\frac{(\cos x)(\cos x)-(\sin x)(-\sin x)}{(\cos x)^2}=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x$$

(4) $f(x)=\sec x=\dfrac{1}{\cos x}$

$$f'(x)=\frac{(\cos x)(0)-1(-\sin x)}{(\cos x)^2}=\frac{\sin x}{\cos^2 x}=\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}=\sec x\tan x$$

(5) $f(x)=\csc x=\dfrac{1}{\sin x}$

$$f'(x)=\frac{(\sin x)(0)-1(\cos x)}{(\sin x)^2}=\frac{-\cos x}{\sin^2 x}=\frac{-1}{\sin x}\cdot\frac{\cos x}{\sin x}=-\csc x\cot x$$

(6) $f(x)=\cot x=\dfrac{\cos x}{\sin x}$

$$f'(x)=\frac{(\sin x)(\sin x)-(\cos x)(\cos x)}{(\sin x)^2}=\frac{-\cos^2 x-\sin^2 x}{\sin^2 x}=\frac{-1}{\sin^2 x}=-\csc^2 x$$

C. Composition and the generalized derivative rules

(1) $f(x)=(g\circ k)(x)=g(k(x))$

$$f'(x)=\lim_{b\to x}\frac{f(b)-f(x)}{b-x}=\lim_{b\to x}\frac{g(k(b))-g(k(x))}{b-x}=\lim_{b\to x}\frac{g(k(b))-g(k(x))}{b-x}\cdot$$

$$\frac{k(b)-k(x)}{k(b)-k(x)}=\lim_{b\to x}\frac{g(k(b))-g(k(x))}{k(b)-k(x)}\cdot\lim_{b\to x}\frac{k(b)-k(x)}{b-x}=$$

$$\lim_{k(b)\to k(x)}\frac{g(k(b))-g(k(x))}{k(b)-k(x)}\cdot\lim_{b\to x}\frac{k(b)-k(x)}{b-x}=g'(k(x))\cdot k'(x)$$

This derivative rule for the composition of functions is called the Chain Rule.

(2) Suppose that $f(x)=g(k(x))$ where $g(x)=x^n$. Then $f(x)=[k(x)]^n$.

$$g(x) = x^n \Rightarrow g'(x) = nx^{n-1} \Rightarrow g'(k(x)) = n[k(x)]^{n-1}.$$

Thus, $f'(x) = g'(k(x)) \cdot k'(x) = n[k(x)]^{n-1} \cdot k'(x)$.

This derivative rule for the power of a function is called the Generalized Power Rule.

2.1 Limits and Continuityod a function of a single variable

Limit – used to describe the way a function varies.

a) Some vary continuously – small changes in x produce small changes in f(x)

b) others vary erratically or jump

c) is fundamental to finding the tangent to a curve or the velocity of an object

Average Speed during an interval of time = distance covered/the time elapsed (measured in units such as: km/h, ft/sec, etc.)

$$= \left(\frac{\Delta \text{ distance}}{\Delta \text{ time}}\right)$$

1.free fall = (discovered by Galileo) a solid object dropped from rest (not moving) to fall freely near the surface of the earth will fall a distance proportional to the square of the time it has been falling $y = 16t^2$ y is the distance fallen after t seconds, 16 is constant of proportionality

Example: A rock breaks loose from a cliff, what is the average speed

a) during first 4 seconds of fall

b) during the 1 second interval between 2 sec. and 3 sec.

a) $\frac{\Delta y}{\Delta x} = \frac{16\,(4)^2 - 16\,(0)^2}{4-0} = \frac{256}{4}$ = 64ft/sec

b) $\frac{\Delta y}{\Delta x} = \frac{16(3)^2 - 16(2)^2}{3-2}$ = 80 ft/sec

2.1.1 Average Rates of Change and Secant Lines:

Find by dividing the change in y by the length of the interval:

Average rate of change of y = f(x) with respect to x over interval $[x_1, x_2]$

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{f(x_1 + h) - f(x_1)}{h} \quad h \neq 0$$

Note: Geometrically the rate of change of over the above interval is the slope of the line through two point of the function (curve) = Secant

2.1.2 LIMITS: Let f(x) be defined on an open interval about c, except possibly at c itself. if f(x) gets very close to L, for all x sufficiently close to c we say that f(x), approaches the limit L written as:

$\lim_{x\to c} f(x)$ =L the limit of f(x) approaches c = L

Example:1 Suppose you want to describe the behaviour of: when x is very close to 4. f(x) $=\frac{.1x^4 - .8x^3 + 1.6x^2 + 2x - 8}{x-4}$

a) First, the function is not defined when x = 4

b) To see what happens to the values of f(x) when x is very close to 4, observe the graph of the function in the viewing window 3.5≤x≤4.5 and 0≤y≤3 -- use the trace feature to move along the graph and examine. The values of f(x) as x get closer to 4

c) Also, notice the "hole" at 4

d) The exploration and table show that as x gets closer to 4 from either side (+/-) the corresponding values of f(x) get closer and closer to 2.

Therefore, the limit as x approaches 4 = 2, $\lim_{x\to 4} f(x)$ =2

Identity Function of Limits: for every real number c,

$\lim_{x\to c} f(c)$ =c

Ex: $\lim_{x\to 2} x$ =2

2.1.3 Limit of a Constant: if d is a constant then

$\lim_{x\to c} d$=d,$\lim_{x\to 2} 3$ =3

& $\lim_{x\to 15} 4$ =4. Nonexistence of Limits (limit of f(x) as x approaches c may fail to exist it.)

1.f(x) becomes infinitely large or infinitely small as x approaches c from either side ex:$\lim_{x\to 0} \frac{1}{x^2}$

2.f(x) approaches L as x approaches c from the right and f(x) approaches M with M≠L, as x approaches c from the left. $\lim_{x\to 0} \frac{|x|}{x}$

A. Function is not defined when x=0. & according to def. of absolute value, |x| = x when x>0 and |x| = -x when x<0 so 2 possibilities: if x>0 then f(x) = 1 If x<0 then f(x) = -1

B. if x approaches 0 from the right, then corresponding values always are 1

C. if x approaches 0 from the left (-values) then corresponding values are always -1

D. So don't approach the same real # as required by def. of limit –Therefore, the limit doesn't exist

3. f(x) oscillates infinitely many times between numbers as x approaches c from either Side.

Example:1 $\lim_{x\to 0}\frac{\sin\pi}{x}$, the values oscillate between -1 and 1 infinitely many times, not approaching one particular real number – so limit doesn't exist.

2.1.4 Calculating using the Limit Laws:

If L, M, c and k are real numbers and:

$\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$

1. Sum Rule: $\lim_{x\to c}(f+g)(x) = \lim_{x\to c}(f(x)+g(x)) = L$+M
2. Difference Rule: $\lim_{x\to c}(f-g)(x) = \lim_{x\to c}(f(x)-g(x)) = L$-M
3. Product Rule: $\lim_{x\to c}(fg)(x) = \lim_{x\to c}(f(x)g(x)) = L$M
4. Quotient Rule: $\lim_{x\to c}(\frac{f}{g})(x) = \lim_{x\to c}(\frac{f(x)}{g(x)}) = \frac{L}{M}$
5. Constant Multiple Rule: lim (k·f(x)) = k·L = $\lim_{x\to c} Kf(x) = K\lim_{x\to c}(f(x)) = KL$,the limit of constant times a function is the constant times the limit
6. Power Rule: if r and s are integers with no common factors and s≠0 then:

$\lim_{X\to C}\sqrt{f(x)} = \sqrt{L}$

7. If f(x) is a polynomial function and c is any real number, then

$\lim_{X\to C} f(x) = f(c)$

Example: 1 $\lim_{x\to -2}(x^2+3x-6) = \lim_{x\to -2} x^2 + \lim_{x\to -2} 3x - \lim_{x\to -2} 6$ (difference rule)

=lim x · lim x + lim 3 · lim x – lim 6 (product rule)

=lim x · lim x + 3 lim x – 6 (limit of a constant rule)

= (-2) (-2) + 3(-2) – 6 (limit of x/Identity rule)

= -8

Example: 2 $\lim_{x\to 2}\frac{x^3-3x^2+10}{x^2-6x+1}$ (Done in 1 step)

$\frac{2^3-3(2)^2+10}{2^2-6(2)+1} = \frac{6}{-7}$ =-.857

Example: 3 $\lim_{x\to 3}\frac{x^2-2x-3}{x-3}$

Denominator. Is 0 at x=3, so try to simplify

$=\frac{x^2-2x-3}{x-3}$

Cancel out new fraction = x +1

= (3) + 1 = 4

Example: 4 $\lim_{x\to -1}\frac{\sqrt{x^2-8}-3}{x+1}$

$= \lim_{x\to -1}\frac{\sqrt{x^2}-8-3\ \sqrt{x^2}+8+3}{x+1\ (\sqrt{x^2}+8+3\ )}$

$=\frac{(x^2+8)-9}{x+1\ (\sqrt{x^2}+8+3\ )}$

$=\frac{(x+1)(x-1)}{x+1\ (\sqrt{x^2}+8+3\ )}$

$=\frac{(x-1)}{(\sqrt{x^2}+8+3\ )}$

= -1/3

__**Sample Problem1**__**:** Find the limit of $\left\{\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right\}$. Consider $f(x) = \frac{\sin(x)}{x}$ .We know from L 'Hospital's Rule that as x approaches zero, the function approaches the limit value of one. Hence, by item (d) above the sequence converges and has the limit value of one.

__**Sample Problem 2**__: Find the limit of $\left\{\frac{\sin(n)}{n}\right\}$. Here we wish to use item (e) above as the squeeze theorem. It is easy to show that for every value of n, $-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$, and that both the first and third sequences converge and that they both have the limit value of zero. Hence, it follows that $\left\{\frac{\sin(n)}{n}\right\}$ converges and has the limit value of zero.

__2.1.5 Sandwich Theorem__:

Refers to a function f whose values are sandwiched between the values of 2 other functions g and h that have the same limit, L, the values of f must also approach L: Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at x =c itself. Suppose also that:

$\lim_{X \to C} g(x)) = \lim_{X \to C} h(x) = L$ then $\lim_{X \to C} f(x) = L$

Ex. if $\sqrt{5} - 2x^2 \le f(x) \le \sqrt{5} - x^2$ for $-1 \le x \le 1$

find $\lim_{X \to 0} f(x)$ $5 - 2(0)^2 \le f(x) \le \sqrt{5} - (0)^2$ it gives $\sqrt{5} \le f(x) \le \sqrt{5}$

Theorem: If $f(x) \le g(x)$ $\forall$ x in some open interval containing c, except possibly at x = c, itself, and the limits of f and g both exist as x approach c, then: $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$

2.1.6 The Precise Definition of a Limit:

Let f(x) is defined on an open interval about (c), except possibly at (c) itself. We say that the limit of f(x) as x approaches (c) is the number L and write: $\lim_{x \to c} f(x) = L$ for every number $\varepsilon > 0$, $\exists$ a corresponding number $\delta > 0$ such that $\forall$ x

$$0 < |x - c| < \delta \,\&\, |f(x) - L| < \varepsilon$$

ε = indicates how close f(x) should be to the limit (the error tolerance)

δ = indicates how close the c must be to get the L (distance from c)

Example: 1 Prove that the $\lim_{x \to 1} (2x + 7) = 9$

Step 1: c = 1, and L =9 so $0 < |x - 1| < \delta$ and $|(2x+7) - 9| < \varepsilon$

Step 2: In order to get some idea which δ might have this property work backwards from the desired conclusion?

$|(2x+7)-9| < \varepsilon$

$|2x - 2| < \varepsilon$

$|2(x-1)| < \varepsilon$ (factor out common)

$|2|\,|x-1| < \varepsilon$

$2|x-1| < \varepsilon$ (divide by 2)

$= |x-1| < \varepsilon/2$ -- this says that $\varepsilon/2$ would be a good choice for δ

Step 3: go forward: $|x-1| < \varepsilon/2$ (get rid of 2 by multiplying on both sides $2|x-1| < \varepsilon$

$|2||x-1| < \varepsilon$

$|2(x-1)| < \varepsilon$

$|2x-2| < \varepsilon$ (rewrite -2 as 7-9)

$|(2x+7)-9| < \varepsilon$

$|f(x) - 9| < \varepsilon$ $\therefore$ $\varepsilon/2$ has required property and proven

2.1.7 Finding δ algebraically for given epsilons

The process of finding a $\delta>0$ such that for all x:

$0<|x-c|<\delta$ ----- $|f(x) - L|<\varepsilon$ can be accomplished in 2 ways:

1. Solve the inequality $|f(x) - L|<\varepsilon$ to find an open interval (a, b) containing x_0 on Which the inequality holds for all $x\neq c$
2. Find a value of $\delta>0$ that places the open interval $(c-\delta, c+\delta)$ centred at x_0 inside the interval (a, b). The inequality $|f(x)-L|<\varepsilon$ will hold for all $x\neq c$ in this δ-interval

Example: 1 Find a value of $\delta>0$ such that for all x, $0<|x-c|<\delta$ ---- $a<x<b$

If a=1 b=7 c=2 so $1<x<7$

Step 1: $|x-2|<\delta$ --- $-\delta<x-2<\delta$ --- $-\delta+2<x<\delta+2$

Step 2: a) $-\delta+2=1$ $-\delta=-1$ --- $\delta = 1$

b) $\delta+2 = 7$ $\delta=5$ closer to an endpoint therefore the value of δ which assures $|x-2|<\delta$ $1<x<7$ is smaller value $\delta=1$

Example: 2 Find an open interval about c on which the inequality $|f(x) - L|<\varepsilon$ holds. Then give a value for $\delta>0$ such that for all x satisfying $0<|x-c|<\delta$ the inequality $|f(x)-L|<\varepsilon$ holds.

If $f(x)=\sqrt{x}$, $L=\frac{1}{2}$ $c=\frac{1}{4}$ $\varepsilon=0.1$

Step 1: $|\sqrt{x} -\frac{1}{2}|< 0.1$ implies $- 0.1<\sqrt{x} - \frac{1}{2}< 0.1$ implies $0.4<\sqrt{x}<.6$ implies $0.16<x<.36$

Step 2: $0<|x-\frac{1}{4}|<\delta = -\delta<x - \frac{1}{4}<\delta = -\delta+\frac{1}{4}<x<\delta+\frac{1}{4}$

a) $-\delta+\frac{1}{4} =.16$ $-\delta.=-09$ -- $\delta=.09$

b) $\delta+\frac{1}{4}=.36$ --- $\delta= .11$ Therefore, $\delta=.09$

Example:3 With the given f(x), point c and a positive number ε, Find $L = \lim_{x\to x_0} f(x)$ then find a number $\delta>0$ such that for all x.

$f(x)=-3x-2$ $x_0=-1$ $\varepsilon=.03$, lim (-3x-2)

$= (-3)(-1)-2 = 1$

Step 1: $|f(x)-L|<\varepsilon$ $= |(-3x-2)-1|<.03$

$= -.03<-3x-3<.03$

$= -1.01<x<-.99$

Step 2: $|x-x_0|<\delta = |x-(-1)| <\delta = -\delta<x+1<\delta = -\delta-1<x<\delta-1$

a) $-\delta-1 =-1.01$ distance to nearer endpoint of $-1.01 = .01$

b) $\delta-1=-.99$ distance to nearer endpoint of $-.99 =.01$ therefore: $\delta=.01$

2.1.8 One-Sided Limit – a limit if the approach is only from one side:

A. Right-hand limit = if the approach is from the right

$\lim_{x \to c} f(x) = L$ Where x > c

B. Left-hand limit = if the approach is from the left

$\lim_{x \to c^-} f(x) = L$ Where x < c

All properties listed for two sided limits apply for one side limits also.

Two Sided Limit Theorem; a function f(x) has a limit as x approaches c if and only if it has left-handed and right hand limits there and the one sided limits equal: $\lim_{x \to c} f(x) = L$ if and only if: $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c} f(x) = L$

2.1.9 Precise Definitions of Right Hand and Left Hand Limits:

f(x) has right hand limit at x_0(c) and write: $\lim_{x \to x_0} f(x) = L$;if for every number ε>0 there exists a corresponding number δ>0, such that ∀ x ,

$x_0 < x < x_0 + \delta \rightarrow |f(x) - L| < \varepsilon$, f(x) has left hand limit at x_0(c) and write

if for every number ε>0 there exists a corresponding number δ>0 such that for all x $x_0 - \delta < x < x_0 \rightarrow |f(x) - L| < \varepsilon$

Theorem – In radian measure its limit as Θ→0 = 1 so… $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ =1 (Θ in radians), finite

Limits as $x \rightarrow \pm\infty$ (have outgrown their finite bounds)

Definition: Limit as x approaches ∞ or -∞:

1. Say f(x) has the limit L as x approaches infinity and write: $\lim_{x \to \infty} f(x) = L$,if, for every number ε>0, there exists a corresponding number M such that for all x: x>M

2. Say f(x) has the limit L as x approaches minus infinity and write: $\lim_{x \to -\infty} f(x) = L$, if for every number ε > 0, there exists a corresponding number N such that for all x: x < N

2.2.10 Properties of Infinite Limits

1. $\lim_{x \to \infty} k = k$ Constant function

2. $\lim_{x \to \pm\infty} \frac{1}{x} = 0$ Identity function

3. Sum, Difference, Product, Constant Multiple, Quotient, and Power Rule all the same with infinity limits as with regular limits.

2.2.11 Limits of Rational Functions:

Divide the numerator and denominator by the highest power of x in the denominator. What happens depends then on the degree of the polynomial:

Example: 1 Find $\lim_{x=0^+}(\frac{1}{3x})$

$$\lim_{x=0^+}(\frac{1}{3x}) = \lim_{x=0^+}(\frac{1}{3x}) = -\infty$$

$$\lim_{x=0^+}(\frac{1}{3x})$$

So $\lim_{x=0^+}(\frac{1}{3x})$does not exist because the limits are not the same

2.2 Continuity:

Definition:

A function g is continuous at a, if $\lim_{x \to a} g(x)$ = g(a). A function is continuous if it is continuous at every a, in its domain. Note that when we say that a function is continuous on some interval it is understood that the domain of the function includes that interval.

For (**example**) the function f(x) = 1-x^2 is continuous on the interval

1 < x < 5 but is not continuous on the interval 1 < x < 1.

Continuous – if you can draw a graph of f(x) at or a certain point without lifting your pencil.

Discontinuous – anytime there is a break, gap or hole at a point in the curve

a) point of discontinuity – the point where the gap/jump is

Right-Continuous – continuous from the right – at a point x=c in its domain if

$$\lim_{x \to c^+} f(x) = f(c)$$

Left-Continuous – continuous from left- at a point c if, $\lim_{x \to c^-} f(x) = f(c)$

2.2.1 Continuity at a point:

1 At an Interior Point – if function y = f(x) is continuous on interior point c of its domain if $\lim_{x \to c^+} f(x) = f(c)$

2. At an Endpoint – y=f(x) is continuous at a left endpoint a, or at right endpoint b, if: $\lim_{x \to a^+} f(x) = f(a)$ (or) $\lim_{x \to b^+} f(x) = f(b)$

Example 1: *Without graphing, show that the function is* $f(x) = \frac{(2-x)\sqrt{2x}}{x^2}$*continuous at x = 3*

Step 1: show $f(3) = \frac{(2-3)\sqrt{2*3}}{3^2} = \frac{\sqrt{6}}{-9}$

Step 2: show $\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{\sqrt{2x}(2-x)}{x^2}$ = limit of quotient $\frac{lim\sqrt{2x}(2-x)}{limx^2}$

$= \frac{lim\sqrt{2x}\ (lim\,(2-x))}{limx^2}$ = limit of a product

$= \frac{\sqrt{lim 2x}(\lim(2-x)}{limx^2}$ =limit of a root $= \frac{(-1)\sqrt{6}}{9}$

$= \frac{\sqrt{6}}{-9}$

So lim f(x) = f (3) and is continuous at x = 3

2.2.2 Definition of Continuity/Continuity Test:

A function f(x) is continuous at x = c if and only if it meets the following 3 conditions:

1. f(c) exists – c lies in the domain of f
2. $\lim_{x\to c} f(x)$ Exists (f has a limit as x approaches c)
3. $\lim_{x\to c} f(x) =$ = f(c) (the limit equals the function value)

2.2.3 Continuity of Special Functions:

a) Every polynomial function is continuous at every real number.

b) Every rational function is continuous at every real number in its domain.

c) Every exponential function is continuous at every real number.

d) Every logarithmic function is continuous at every positive real number.

e) F(x) = sin x and g(x) = cos x are continuous at every real number.

f) H(x) = tan x is continuous at every real number in its domain.

2.2.4 Continuity on the Interval:

A function is continuous on the interval if and only if it is continuous at every point of the interval [a, b]. If the function is continuous on the closed interval [a, b] provided that f is continuous from the right at x= a and from the left at x=b and continuous at every value in the open int. (a, b).

2.2.5 Properties of Continuty functions:

If the functions f and g are continuous at x=c, then the following combinations are continuous at x = c.

1. Sums: f + g
2. Differences: f-g

3. Products: $f \cdot g$
4. Constant Multiples: $k \cdot f$ for any number k
5. Quotients: $\frac{f}{g}$ provided $g(c) \neq 0$
6. Powers: $f^{\frac{r}{s}}$ provided it is defined on the open interval containing c, and r, s is integer.

Example: 1 Show that $h(x) = \sqrt{x^3 - 3x^2 + x + 7}$ is continuous at x = 2

Steps: first show $f(2) = 2^3 - 3(2)^2 + 2 + 7 = 5$, Then check $g(x) = \sqrt{x}$ which is continuous b/c by power property $\sqrt{\lim_{x \to 5} x} = \sqrt{5}$,

So, with c=2 and f(c) =5, the composite function g∘f given by:

$(g \circ f)(x) = (g(f(x)) = g(x^3 - 3x^2 + x + 7) = \sqrt{x^3 - 3x^2 + x + 7}$

2.2.6 Continuous Extension to a Point:

Often a function (such as a rational function) may have a limit even at a point where it is not defined.

if f(c) is not defined, but $\lim_{x \to c} f(x) = L$ exists, a new function rule can be defined as:

$$f(x) = f(x) \quad \text{if x is in the domain of f}$$
$$= L \quad \text{if } x = c$$

In rational functions, f, continuous extensions are usually found by cancelling common factors.

Example: 1 Show that $f(x) = \frac{x^2 + x - 6}{x^2 - 4}$ has a continuous extension to x=2, find the extension

First factor $\frac{(x-2)(x+3)}{(x-2)(x+2)} = \frac{(x+3)}{(x+2)}$ which is equal to f(x) for $x \neq 2$, but is continuous at x=2 shows continuous by plugging into new function$= \frac{(2+3)}{(2+2)} = \frac{5}{4}$ have removed the point of discontinuity at 2.

2.2.7 Intermediate Value Theorem for Continuous Functions:

A function y = f(x) that is continuous on a closed interval [a,b] takes on every value between f(a) and f(b). In other words, if y_0 is any value between f(a) and f(b) then $y_0 = f(c)$ for some c in [a,b]

- What this is saying Geometrically is that – any horizontal line $y = y_0$ crossing the y-axis between the numbers f(a) and f(b) will cross the curve y=f(x) at least once over the interval.

- For this theorem-the curve must be continuous with no jumps/breaks.
- This theorem tells us that if f is continuous, then any interval on which f changes signs contains a zero/ root of the function.

2.2.8 Tangents and Derivatives:

We will now study how to find the tangent of an arbitrary curve at point $P(x_0, f(x_0))$

To do this we must:

1. Calculate the slope of the secant through P and a point $Q(x_0+h, f(x_0+h))$
2. Then investigate the limit of the slope as h approaches 0
   a) if limit exists—we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope
   b) The slope of the curve $y=f(x)$ at the point $P(x_0, f(x_0))$ is the following:

$$m = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{(Provided the limit exists)}$$

The tangent line to the curve at P is the line through P with this slope.

$y=y_0 + m(x-x_0)$

2.2.9 Difference Quotient of F:

$\frac{f(x_0+h) - f(x_0)}{h}$ has a limit as h approaches 0 called the derivative of f at x_0

1) if interpreted as the secant slope—the derivative gives the slope of the curve and tangent at the point where $x=x_0$
2) if interpreted at the average rate of change– the derivative gives the function's rate of change with respect to x at $x=x_0$

Example: 1 Find an equation for the tangent to the curve at the given point. Then sketch the curve and tangent together.

$y= (x-1)^2 +1$ at pt (1, 1)

$$= \lim_{x \to 0} \frac{[(1+h-1)^2+1- [(1-1)^2+1]}{h}$$

$$= \lim_{x \to 0} \frac{h^2}{h}$$

= lim h = 0 (b/c constant), so at (1,1) $y=1+0(x-1)$, $y=1$ is tangent line

Example: 2 Find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

$F(x) \quad = x-2x^2$ (1, -1)

$$= \lim_{x\to 0} \frac{(1+h)-2(1+h)^2]-[1-2(1)^2]}{h}$$

$$= \lim_{x\to 0} \frac{(1+h-2-4h-2h^2)+1}{h} = -3$$

At (1, -1) = y +1 = -3(x-1)

2.3 Rolle's theorem:

If f is a function such that

(i) f is continuing in the closed interval [a, b]

(ii) f' (x) exists for every point x in the open interval (a, b) or] a,b[

(iii) f (a) = f (b) there is a point c where $a < c < b \ni f'$ (c) = 0

Proof: The function f, being conditions in the closed interval [a, b] is bounded and attains its least upper bound and greatest lower bound. Let M, m be the least upper bound and greatest lower bound of, f respectively and it can be such that

f (c) = M, f (d) = m either M=m or M≠m

Now M=m implies f (x) = M $\forall\, x \in [a, b]$, implies f '(x) = 0 $\forall\, x \in [a, b]$

Thus, the theorem is true in this case, now suppose that M≠m, as f (a) = f (b) and M≠m atleast one of the numbers M and m must be different from f (a) and f (b). Let M be different from each of f (a) and f (b) we have M= f (c),

M≠f (a), M≠f (b), Now f (c) ≠f (a), implies c≠a

f (c) ≠f (b), implies c≠b, thus a <c < b. The function is derivable at c. We shall show that

f '(c) $= 0$, If f '(c)>0, there exists δ> 0 such that

f (x) >f (c) =M 0 $\forall\, x \in]c, c+\delta]$. But M being the least upper bound, we have

f (x) ≤ M $\forall\, x \in [a, b]$

Thus, we have a contradiction we cannot have f '(c) > 0,

Now suppose that f '(c) <0, so that there exists δ >0 such that

f (x) >f (c)=M $\forall\, x \in [c-\delta, c[$.

This again is not possible. Thus, we cannot have f '(c) <0. We conclude that

f '(c)=0.

Problem: 1

Verify Rolle's Theorem for the following function

f(x) = 2x^3+x^2-4x -2 in [-$\sqrt{2}, \sqrt{2}$]

Solution:

Since f is a rational integral function of x it is continuous and differentiable for all real values of x. Hence the first two condition of Rolle's Theorem are satisfied in any interval in order to find the interval.

Let f(x) =0

$2x^3+x^2$-4x -2=0

$x^2(2x+1)$-2(2x +1) =0

$(2x+1)(x^2-2)=0$

$(2x+1)$ =0 and $(x^2-2)=0$

x=$-\frac{1}{2}$ x =$\pm\sqrt{2}$

f($\sqrt{2}$)=- f($\sqrt{2}$)=f(-1) consider the interval [-$\sqrt{2},\sqrt{2}$] all the conditions of rolle's theorem is satisfied to verify the 3^{rd} condition obtain f'(x), f '(x)=6x^2+2x-4 implies 6x^2+2x-4 =0. Equating it to zero, we get value of x as

6x^2+2x-4 =0,

3x^2+x-2 =0,

(x+1) (x-$\frac{2}{3}$) =0

x =-1, x=$\frac{2}{3}$

Where f' (-1) =6$(-1)^2$+2(-1)-4 =0,

f' ($\frac{2}{3}$)=6$(\frac{2}{3})^2$+2($\frac{2}{3}$)-4 =0, Since both the points x=-1 and x=$\frac{2}{3}$ lie in the open interval [-$\sqrt{2},\sqrt{2}$] Rolle's theorem is satisfied.

2.4 Mean Value Theorem: If two functions F and f is

i) Continuous in the interval [a, b]

ii) Derivable in the interval $]a,b[$

iii) F'(x) $\forall\ x \in\ ]a,b[$ then there exists one point c $\in\ ]a,b[$

Such that $\frac{f(b)-f(a)}{F(b)-F(a)}=\frac{f'(x)}{F'(x)}$

Proof: Let a function ϕ can be defined by ϕ (x) =f(x)+ A F(x) where A is a constant, to be determined such that

$[f(b) - f(a)]$ A = $-[f(b) - f(a)]$ ... ... (1)

ф (a) = ф (b) requires F (b) – F(a) ≠ 0 if it is zero, then functions F would satisfies all the conditions of Rolle's theorem A= $-\frac{[f(b)-f(a)]}{[F(b)-F(a)]}$ A function ф is continuous in the [a, b] derivable in the] a, b[and ф (a) = ф (b).Hence by Rolle's theorem. For all there exist a point c belongs to] a, b [such that ф'(c) = 0.

ф (x) = f (x) +A F (x)

ф'(x) = f (x) + A F'(x)

At x = c,

Φ' (c) = f '(c) +A F '(c)

o = f '(c) + A F' (c)

$$\frac{f'(C)}{F'(C)} = -A$$

$\frac{f(b)-f(a)}{F(b)-F(a)} = \frac{f'(c)}{F'(c)}$ using (1) F'(c) ≠ 0

2.5 Taylor's Theorem:

If f is a real valued function on [a, a+h] ∋ all the derivatives upto $(n-1)^{th}$ are continuous in a$\leq x \leq a+h$ and $f_{(x)}^{(n)}$ exists in a$\leq x \leq a+h$ then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots . \frac{n^n}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h),$$

$0 < \theta < 1.$

Proof: p is a given positive integer, then there exists, at least one number, θ between 0 and 1 such that

$$F(a+h) = f(a) + h\, f'(a) + \frac{h^2}{2!} + f''(a) + \cdots \frac{n^n}{(n-1)!}(1-\theta)^{n-1} f^n(a+\theta h) \;......(1)$$

The above equation implies the continuity of each off, f ',f``.......f^{n-2} in the closed interval [a, a+h].

Let a function ф be defined by

$$f(a+h) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \cdots . \frac{(a+h-x)^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{n-1}(x) + A(a+h-x)^p$$

Here A is a constant to be determined such that (a) = ф(a+h) thus a is given by

$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots \frac{n^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{n-1}(a) + Ah^p$ ....(2)

The function ϕ is continuous in the closed interval [a, a+h], derivable in the open interval] a, a+h [and $\phi(a) = \phi(a+h)$. Hence by rolle's theorem, there exists atleast one number, θ between 0 and 1 such that $\phi'(a+h)$ =0 but $\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x)$ - $Ap(a+h-x)^{p-1}$

$0 = \phi'(a+\theta h) = \frac{(h)^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h) - Ap(1-\theta)^{n-1}(h)^{p-1}$

Implies $A = \frac{(h)^{n-p}(1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h)$, for $(1-\theta) \neq 0$ and $h \neq 0$

Substituting the values of A in the required result (1)

i. Reminder after n terms, the term $R_n = \frac{(h)^n(1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h)$, is known as Taylor'form of Remainder R_n after n terms and is due to Schlomileh and Roche.

ii) Putting p = 1, we obtain $R_n = \frac{(h)^n(1-\theta)^{n-p}}{p(n-1)!} f^n(a+\theta h)$, which form of reminder is due to Cauchy.

iii) Putting p = n, we obtain $R_n = \frac{(h)^n}{n!} f^n(a+\theta h)$, which is due to Lagrange.

Example:1 $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x+\theta h)$,find the value of θ as $x \to a$ if $f(x) = ((x-a)^{\frac{5}{2}})$

Solution: $f(x) = ((x-a)^{\frac{5}{2}})$,

$f'(x) = \frac{5}{2}((x-a)^{\frac{3}{2}})$,

$f''(x) = \frac{5}{2}(\frac{3}{2}(x-a)^{\frac{1}{2}}) = \frac{15}{4}(x-a)^{\frac{1}{2}})$

$f''(x+\theta h) = \frac{15}{4}(x-a)^{\frac{1}{2}})$ substituting expression

$f(x+h) = ((x-a)^{\frac{5}{2}}), +h\frac{5}{2}((x-a)^{\frac{3}{2}})) + \frac{h^2}{2!}\frac{15}{4}(x-a)^{\frac{1}{2}})$ when $x \to a$ we get

$h^{\frac{5}{2}} = 0 + 0 + \frac{15}{4}\frac{h^2}{2!}(\theta h)^{\frac{1}{2}}$

$h^{\frac{5}{2}} = \frac{15h^2}{8}(\theta h)^{\frac{1}{2}}\theta = \frac{65}{225}\theta \in (0,1)$. Therefore, the Taylor's theorem is verified

UNIT - III

INTEGRABLE FUNCTIONS:

The process of finding antiderivatives is called ***antidifferentiation***, more commonly referred to as ***integration.*** We have a particular sign and set of symbols we use to indicate integration:

$$\int f(x)dx = F(x) + C.$$

We refer to the left side of the equation as “the indefinite integral of $f(x)$ with respect to x." The function $f(x)$ is called the ***integrand*** and the constant C is called the ***constant of integration.*** Finally the symbol dx indicates that we are to integrate with respect to x.

Using this notation, we would summarize the last example as follows:

$$\int 3x^2 dx = x^3 + C$$

Using Derivatives to Derive Basic Rules of Integration

As with differentiation, there are several useful rules that we can derive to aid our computations as we solve problems. The first of these is a rule for integrating power functions, $f(x) = x^n \; [n \neq -1]$,and is stated as follows:

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C.$$

We can easily prove this rule. Let $F(x) = \frac{1}{n+1}x^{n+1} + C, n \neq -1$. We differentiate with respect to $x$and we have:

$$\begin{aligned}
F'(x) &= \frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) = \frac{d}{dx}\left(\frac{1}{n+1}x^{n+1}\right) + \frac{d}{dx}(C) \\
&= \left(\frac{1}{n+1}\right)\frac{d}{dx}\left(x^{n+1}\right) + \frac{d}{dx}(C) \\
&= \left(\frac{n+1}{n+1}\right)x^n + 0 \\
&= x^n.
\end{aligned}$$

The rule holds for $f(x) = x^n[n \neq -1]$.What happens in the case where we have a power function to integrate with $n = -1$,say $\int x^{-1}dx = \int \frac{1}{x}dx$. We can see that the rule does not work since it would result in division by 0. However, if we pose the problem as finding $F(x)$ such that $F'(x) = \frac{1}{x}$, we recall that the derivative of logarithm functions had this form. In particular, $\frac{d}{dx}\ln x = \frac{1}{x}$. Hence

$$\int \frac{1}{x} dx = \ln x + C.$$

In addition to logarithm functions, we recall that the basic exponentional function, $f(x) = e^x$, was special in that its derivative was equal to itself. Hence, we have

$$\int e^x dx = e^x + C.$$

Again, we could easily prove this result by differentiating the right side of the equation above. The actual proof is left as an exercise to the student.

As with differentiation, we can develop several rules for dealing with a finite number of integrable functions. They are stated as follows:

If f and g are integrable functions, and C is a constant, then

$$\begin{aligned} \int [f(x) + g(x)]\, dx &= \int f(x) dx + \int g(x) dx, \\ \int [f(x) - g(x)] dx &= \int f(x) dx - \int g(x) dx, \\ \int [C f(x)] dx &= C \int f(x) dx. \end{aligned}$$

Example 2:

Compute the following indefinite integral.

$$\int \left[2x^3 + \frac{3}{x^2} - \frac{1}{x} \right] dx.$$

Solution:

Using our rules, we have

$$\begin{aligned} \int \left[2x^3 + \frac{3}{x^2} - -\frac{1}{x} \right] dx &= 2 \int x^3 dx + 3 \int \frac{1}{x^2} dx - \int \frac{1}{x} dx \\ &= 2 \left(\frac{x^4}{4} \right) + 3 \left(\frac{x^{-1}}{-1} \right) - \ln x + C \\ &= \frac{x^4}{2} - \frac{3}{x} - \ln x + C. \end{aligned}$$

Sometimes our rules need to be modified slightly due to operations with constants as is the case in the following example.

Example 3:

Compute the following indefinite integral:

$$\int e^{3x} dx.$$

Solution:

We first note that our rule for integrating exponential functions does not work here since $\frac{d}{dx}e^{3x} = 3e^{3x}$. However, if we remember to divide the original function by the constant then we get the correct antiderivative and have

$$\int e^{3x} dx = \frac{e^{3x}}{3} + C.$$

We can now re-state the rule in a more general form as

$$\int e^{kx} dx = \frac{e^{kx}}{k} + C.$$

Differential Equations

We conclude this lesson with some observations about integration of functions. First, recall that the integration process allows us to start with function f from which we find another function $F(x)$ such that $F'(x) = f(x)$. This latter equation is called a ***differential equation.*** This characterization of the basic situation for which integration applies gives rise to a set of equations that will be the focus of the Initial Value Problem.

Example 4:

Solve the general differential equation $f'(x) = x^{\frac{2}{3}} + \sqrt{x}.$

Solution:

We solve the equation by integrating the right side of the equation and have

$$f(x) = \int f'(x)dx = \int x^{\frac{2}{3}} dx + \int \sqrt{x} dx.$$

We can integrate both terms using the power rule, first noting that $\sqrt{x} = x^{\frac{1}{2}}$, and have

$$f(x) = \int x^{\frac{2}{3}} dx + \int x^{\frac{1}{2}} dx = \frac{3}{5} x^{\frac{5}{3}} + \frac{2}{3} x^{\frac{3}{2}} + C.$$

3.1 Riemann Integration:

In elementary calculus, the process of integrations treated as the inverse operation of differentiation and the integral of the function is called an anti-derivative. The definite integral is given by Germany mathematician Riemann (1820-1866) in this concept dealing with closed finite intervals [a,b] so that (b-a)$\in$ r implies

a$\leq x \leq b$ more over all function f will be assume to be a real valued functions defined and bounded on [a,b].Thus symbolically f(a,b) and $|f(x)| \leq k$ where k is a positive real number.

3.1.1 Definition of partition of closed interval:

Let I =[a,b] be a finite closed interval a<x_0<$x_1 < x_2 < \cdots \ldots < x_n < b$ is the finite ordered set P=$\{x_0$,$x_1, x_2, \ldots \ldots, x_n\}$ is called a partition of I, the (n+1) points x_0 ,$x_1, x_2, \ldots \ldots, x_n$ are called partition points of P. The n closed sub intervals $I_1 = [x_0, x_1]; I_2 = [x_2, x_3] I_3 = [x_0, x_1], \ldots \ldots I_n = [x_{n-1}, x_n]$ (i.e) $\bigcup_{r=1}^{n} I_r$=$(x_{r-1,} x_n)$ = [a, b] = I, Where $I_1 \ldots I_2 \ldots \ldots \ldots I_n$ are called the segments of partition of p.

3.1.2 Norm of a Partition:

The maximum of the length of the sub intervals of a partition p is called the Norm or Mesh of the partition p and denoted by $\|P\|$

3.2 Definition of Reimann Integral:

A bounded function f is said to be Riemann integral function or R- integral on [a,b] if its lower and upper Riemann integrals are equal.

3.3 Darboux's Theorem:

Let f be a bounded function on the closed interval [a, b] then given any ε>0,$\exists$ a $\delta > 0, such that$ for all partition P with $\|P\|$<δ

U (P, f) <$\int_a^b f(x)dx + \varepsilon$ and L(P, f)>$\int_a^b f(x)dx - \varepsilon$

Proof: Be definition of if there exists a partition P_1 such that ε >0,

U (P, f) <$\int_a^b f(x)dx + \frac{\varepsilon}{2}$. Let P_1 has K points other than the end points a and b. we may assume that K$\geq$ 1 if possible by allowing refinement of P_1.

Let $\delta = \frac{\varepsilon}{2(M-m)K}$. Let P be a partition with $\|P\| < \delta_1 < \delta$. We will show that the conclusion of the theorem holds for this partition P. Let P_2 be the comment refinement of P_1 and P. Let P_2 has r more points than P. We see that these points are points of P_1 and as P_1 has k points other than end points we have 0 < r < K. Now we have 0$\leq$ U (P,f) - L(P,f)$\leq (M-m)r\,\delta_1$................(1)

and also we have U(P,f) $\leq$ L(P,f)$\int_a^b f(x)dx + \frac{\varepsilon}{2}$..................(2)

Combining (1) and (2) we get U(P,f) $< \int_a^b f(x)dx + \frac{\varepsilon}{2} + (M-m)r\,\delta_1$

$\leq \int_a^b f(x)dx + \frac{\varepsilon}{2} + (M-m)\,\delta K$ as (0$\leq$ r $\leq$ K and $\delta_1 < \delta$)

$\leq \int_a^b f(x)dx + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \int_a^b f(x)dx + \varepsilon$

Example:1 consider the function f(x) = x in the interval [0.1], we have that f$\in$R (0,1) and $\int_a^b f(x)dx = \frac{1}{2}$

For any positive integer n we consider the partition P_n of [0.1] in $\{0, \frac{1}{n}, \frac{2}{n^2}, \frac{3}{n^3} \dots \dots \frac{n-1}{n}, 1\}$.

As the function is monotone increasing in $\{\frac{r-1}{n}, \frac{r}{n}\}$ we have M=$\frac{r}{n}$, m =$\frac{r-1}{n}$

Also, we have $\delta_r = \frac{1}{n}$. Hence U (P, f) =$\sum_{r=1}^{n} \frac{r}{n} \quad \frac{1}{n} = \sum_{r=1}^{n} \frac{r}{n^2} \quad \{1 + 2 \dots . + n\}$

3.4 Fundamental theorem of integral calculus:

Statement: let f be a continuous function defined on [a, b] and ϕ be a differential function on [a, b] such that ϕ'(x)= f(x) $\forall x \in$[a, b] then $\int_a^b f_1\ (x)$ dx=ϕ(b) ϕ(a)

Proof: Let F=$\int_a^b f\ (x)$ dx and F'(x) = f(x) $\forall x \in$[a, b] also given that ϕ(x)= f(x), F'(x)= ϕ'(x)

F'(x)- ϕ'(x) =0, implies [F'- ϕ']x=0

F'- ϕ'=c, c is constant

f (x) $\neq \phi(x)$+c

f (a)$= \phi(a)$+c, f(b)$= \phi(b)$+c but from the definition

F(x)=$\int_a^b f\ (x)$ dx =0

F (b) - F(a)= $\phi(b)$+c- $\phi(a)$+c, $\phi(b) = \phi(a)$=0

$\int_a^b f(x)\,\mathrm{dx} = \phi(\mathrm{b}) - \phi(\mathrm{a})$

3.5 First mean Value Theorem:

Statement: Let $'f\,'$ be a reimann integral on [a, b] then $\exists$ 'μ' lies between

$\int_a^b \delta = \mu$(b-a)

Proof: From the definition of Riemann sums

$m(b-a) \le$ L(P,f) $\le$ U(P,f) $\le \mu(b-a)$

$m(b-a) \le$ L(P,f) $\le \mu(b-a)$ taking supremimum on the above inequality we get,

$m(b-a) \le$ Sup L(P ,f) $\le \mu(b-a)$

$m(b-a) \le \int_a^b f(x)\,\mathrm{dx} \le \mu(b-a)$ ………(1)

Also, we know that $m(b-a) \le$ U(P, f) $\le \mu(b-a)$ taking infimum on the above inequality we get,

m$(b-a) \le$ inf U (P ,f) $\le \mu(b-a)$

m$(b-a) \le \int_a^b f(x)\,\mathrm{dx} \le \mu(b-a)$ ……..(2). As f is Riemann integral we have

$\int_a^b f(x)\,\mathrm{dx} = \int_a^b f = \int_a^b f$ ……………………..(3)

combining (1) (2) and (3) m$(b-a) \le \int_a^b f \le \mu(b-a)$

$\Rightarrow \int_a^b f = \mu(b-a)$ where μ is the values lies between the bounds m and μ

3.6 Improper Integrals

If the function f becomes unbounded on [a, b] or if the limits of the integration becomes infinite then the symbol

$\int_a^b f(x)\,dx$, is called the improper integral

First kind

If either one or both limits are infinite and the enterable is bounded

Second kind

It the intervals are finite and f becomes unbounded then it is called improper integrals of second kind

3.7 Integral with Infinite Intervals

Definition: If the function f id bounded the inferable for x $\geq$ a then by definition

1. If $\int_a^M f(x)\,dx$ exists for every number

$M \geq a$, then $\int_a^\infty f(x)\,dx = \lim_{M\to\infty} \int_a^M f(x)\,dx$ provided this limit exists (as a finite number).

2. If $\int_M^b f(x)\,dx$ exists for every number $M \leq b$,

Then $\int_{-\infty}^b f(x)\,dx = \lim_{M\to-\infty} \int_M^b f(x)\,dx$ provided this limit exists (as a finite number).

[*Note*: The integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are said to be convergent if the corresponding limit exists and divergent if the limit does not exist.] $\int_a^M f(x)$

Is said to be converge to the value M if given $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$.

[*Note*: Any real number *a* can be used.]

Problems:

1. Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{M\to\infty} \int_1^M x^{-2}\,dx = \lim_{M\to\infty} \left[\frac{-1}{x}\right]_1^M = \lim_{M\to\infty} \left[\frac{-1}{M}+1\right] = 1 \Rightarrow \int_1^\infty \frac{1}{x^2}\,dx \text{ Converges to 2.}$$

2. Evaluate $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{M\to\infty}\int_1^M \frac{1}{\sqrt{x}}\,dx = \lim_{M\to\infty}\left\{2\sqrt{x}\right\}_1^M$$

$$= \lim_{M\to\infty}\left\{2\sqrt{M}-2\right\} = \infty \Rightarrow \int_1^\infty \frac{1}{\sqrt{x}}\,dx \text{ diverges.}$$

3. Evaluate $\int_{-\infty}^0 e^x\,dx$.

$$\int_{-\infty}^0 e^x\,dx = \lim_{M\to-\infty}\int_M^0 e^x\,dx = \lim_{M\to-\infty}\left\{e^x\right\}_M^0$$

$$= \lim_{M\to-\infty}\left\{e^0 - e^M\right\} = 1 - 0 = 1 \Rightarrow \quad \int_{-\infty}^0 e^x\,dx \text{ converges to 1.}$$

4. Evaluate $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx \int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx = 2\int_0^\infty \frac{1}{1+x^2}\,dx$ (by symmetry) =2

$$2\lim_{M\to\infty}\int_0^M \frac{1}{1+x^2}\,dx = 2\lim_{M\to\infty}\left\{\arctan x\right\}_0^M = 2\lim_{M\to\infty}\left\{\arctan M - \arctan 0\right\} =$$

$$2\lim_{M\to\infty}\arctan M = 2\left(\frac{\pi}{2}\right) = \pi \Rightarrow \int_{-\infty}^\infty \frac{1}{1+x^2}\,dx \text{ converges to } \pi.$$

5. Evaluate $\int_1^\infty \ln x\,dx$.

$$\int_1^\infty \ln x\,dx = \lim_{M\to\infty}\int_1^M \ln x\,dx = \lim_{M\to\infty}\left\{x\ln x - x\right\}_1^M = \lim_{M\to\infty}\left[\left\{M\ln M - M\right\} -\right.$$

$$\left.\left\{1\ln 1 - 1\right\}\right] = \lim_{M\to\infty}\left\{M(\ln M - 1)\right\} + 1 = \left\{\lim_{M\to\infty} M\right\}\cdot\left\{\lim_{M\to\infty}(\ln M - 1)\right\} + 1 =$$

$$\infty\cdot\infty + 1 = \infty \Rightarrow \int_1^\infty \ln x\,dx \text{ diverges.}$$

6. Evaluate $\int_0^{\infty} xe^{-x}\ dx$.

$$\int_0^{\infty} xe^{-x}\ dx = \lim_{M\to\infty}\int_0^{M} xe^{-x}\ dx = \lim_{M\to\infty}\left\{-xe^{-x}-e^{-x}\right\}_0^M = \lim_{M\to\infty}\left\{\frac{-x}{e^x}-\frac{1}{e^x}\right\}_0^M =$$

$\lim_{M\to\infty}\left\{\frac{-M}{e^M}-\frac{1}{e^M}\right\}-\left\{\frac{-0}{e^0}-\frac{1}{e^0}\right\}$. By L'Hospital's Rule, $\lim_{M\to\infty}\left\{\frac{-M}{e^M}\right\} =$

$\lim_{M\to\infty}\left\{\frac{-1}{e^M}\right\}=0$. Thus, $\lim_{M\to\infty}\left\{\frac{-M}{e^M}-\frac{1}{e^M}\right\}-\left\{\frac{-0}{e^0}-\frac{1}{e^0}\right\} = (0-0)-(0-1)=1$.

Thus, $\int_0^{\infty} xe^{-x}\ dx$ converges to 1.

3.7.1 Improper Integral with Discontinuous Integral:

Definition

1. If f is continuous on [a, *b*) and is discontinuous at *b*, then $\int_a^b f(x)\ dx =$

$\lim_{M\to b^-}\int_a^M f(x)\ dx$ if this limit exists (as a finite number).

2. If f is continuous on (a, *b*] and is discontinuous at *a*, then $\int_a^b f(x)\ dx =$

$\lim_{M\to a^+}\int_M^b f(x)\ dx$ if this limit exists (as a finite number).

Note: The improper integral $\int_a^b f(x)\ dx$ is called convergent if the corresponding limit exists and divergent if the limit does not exist.

3. If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\, dx$ and $\int_c^b f(x)\, dx$ are convergent, then we define

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Examples:

1. Evaluate $\int_0^1 \frac{1}{x^2}\, dx = \int_0^1 \frac{1}{x^2}\, dx = \lim_{M\to 0^+} \int_M^1 x^{-2}\, dx = \lim_{M\to 0^+} \left\{\frac{-1}{x}\right\}_M^1 = \lim_{M\to 0^+} \left\{-1 + \frac{1}{M}\right\} = +\infty$

$$\Rightarrow \int_0^1 \frac{1}{x^2}\, dx \text{ diverges.}$$

2. Evaluate $\int_0^4 \frac{1}{\sqrt{x}}\, dx$.

$$\int_0^4 \frac{1}{\sqrt{x}}\, dx = \lim_{M\to 0^+} \int_M^4 x^{-1/2}\, dx = \lim_{M\to 0^+} \left\{2\sqrt{x}\right\}_M^4 = \lim_{M\to 0^+} \left\{4 - 2\sqrt{M}\right\} = 4 - 2\sqrt{0} = 4 \Rightarrow \int_0^4 \frac{1}{\sqrt{x}}\, dx$$

Converges to 4.

3. Evaluate $\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$.

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx = \lim_{M\to 1^-} \int_0^M \frac{1}{\sqrt{1-x^2}}\, dx = \lim_{M\to 1^-} \left\{\arcsin x\right\}_0^M =$$

$$\lim_{M\to 1^-} \left\{\arcsin M - \arcsin 0\right\} = \arcsin 1 - 0 = \frac{\pi}{2} \Rightarrow \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx \text{ Converges to } \frac{\pi}{2}.$$

4. Evaluate $\int_0^1 \ln x\, dx$.

$$\int_0^1 \ln x \, dx = \lim_{M \to 0^+} \int_M^1 \ln x \, dx = \lim_{M \to 0^+} \{x \ln x - x\}_M^1 =$$

$$(1\ln 1 - 1) - \lim_{M \to 0^+} (M \ln M - M) = -1 - \lim_{M \to 0^+} (M \ln M) + 0 = -1 + \lim_{M \to 0^+} M \ln M \, .$$

By L'Hospital's Rule,

$$\lim_{M \to 0^+} M \ln M = \lim_{M \to 0^+} \frac{\ln M}{1/M} = \lim_{M \to 0^+} \frac{1/M}{-1/M^2} = 5$$

$\lim_{M \to 0^+} (-M) = 0$. Thus, $\lim_{M \to 0^+} \int_M^1 \ln x \, dx = -1 \Rightarrow \int_0^1 \ln x \, dx$ converges to – 1.

5. Evaluate $\int_1^2 \frac{1}{x \ln x} dx$.

$$\int_1^2 \frac{1}{x \ln x} dx = \lim_{M \to 1^+} \int_M^2 \frac{1}{x \ln x} dx = \lim_{M \to 1^+} \{\ln(\ln x)\}_M^2$$

$$= \lim_{M \to 1^+} \ln(\ln 2) - \lim_{M \to 1^+} \ln(\ln M) = \ln(\ln 2) - \ln(\ln(1)) = \ln(\ln 2) - \ln(0) = \ln(\ln 2) - (-\infty)$$

$\Rightarrow \int_1^2 \frac{1}{x \ln x} dx$ diverges.

6. Evaluate $\int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$.

$$\int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \lim_{M \to 0^+} \int_M^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \lim_{M \to 0^+} \left\{2e^{\sqrt{x}}\right\}_M^4 = \lim_{M \to 0^+} \left\{2e^{\sqrt{4}} - 2e^{\sqrt{M}}\right\}$$

$= 2e^2 - 2e^0 = 2e^2 - 2 \Rightarrow \int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$ Converges to $2e^2 - 2$.

(7) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{M\to 1^-}\int_0^M \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{M\to 1^-}\{\arcsin x\}_0^M = \lim_{M\to 1^-}\{\arcsin M - \arcsin 0\} =$

$\arcsin 1 - \arcsin 0 = \frac{\pi}{2}$.

Thus, $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ converges to $\frac{\pi}{2}$.

(8) $\int_1^\infty \frac{e^{1/x}}{x^2}\,dx = \lim_{M\to\infty}\int_1^M \frac{e^{1/x}}{x^2}\,dx = \lim_{M\to\infty}\left\{-e^{1/x}\right\}_1^M = \lim_{M\to\infty}\left\{-e^{1/M} + e\right\} = -1 + e$.

Thus, $\int_1^\infty \frac{e^{1/x}}{x^2}\,dx$ converges to $e-1$.

(9) $\int_0^4 \frac{1}{\sqrt{4-x}}dx = \lim_{M\to 4^-}\int_0^M \frac{1}{\sqrt{4-x}}\,dx = \lim_{M\to 4^-}\left\{-2\sqrt{4-x}\right\}_0^M = \lim_{M\to 4^-}\{-2\sqrt{4-M}+4\} = 4$.

Thus, $\int_0^4 \frac{1}{\sqrt{4-x}}$ converges to 4.

(10) $\int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{M\to 0^+}\int_M^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{M\to 0^+}\left\{2e^{\sqrt{x}}\right\}_M^4 = \lim_{M\to 0^+}\left\{2e^{\sqrt{4}} - 2e^{\sqrt{M}}\right\} = 2e^2 - 2$.

Thus, $\int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ converges to $2e^2 - 2$.

(11) $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{M\to\infty}\int_e^M \frac{1}{x(\ln x)^2} = \lim_{M\to\infty}\left\{\frac{-1}{\ln x}\right\}_e^M = \lim_{M\to\infty}\left\{\frac{-1}{\ln M}+1\right\} = 1$.

Thus, $\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$ converges to 1.

(12) $\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx = \lim_{M\to 3^-}\int_0^M \frac{x}{\sqrt{9-x^2}}\,dx = \lim_{M\to 3^-}\left\{-\sqrt{9-x^2}\right\}_0^M = \lim_{M\to 3^-}\left\{-\sqrt{9-M^2}+3\right\} = 3$.

Thus, $\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx$ converges to 3.

(13) $\int_0^\infty \frac{x^2}{1+x^6}\,dx = \lim_{M\to\infty} \int_0^M \frac{x^2}{1+x^6}\,dx$

$$= \lim_{M\to\infty}\left\{\frac{1}{3}\arctan(x^3)\right\}_0^M = \lim_{M\to\infty}\left\{\frac{1}{3}\arctan(M^3)\right\} - \quad 0 = \frac{1}{3}\left(\frac{\pi}{2}\right) = \frac{\pi}{6}.$$

Thus, $\int_0^\infty \frac{x^2}{1+x^6}\,dx$ converges to $\frac{\pi}{6}$.

(14) $\int_1^2 \frac{1}{x\sqrt{x^2-1}}\,dx = \lim_{M\to 1^+} \int_M^2 \frac{1}{x\sqrt{x^2-1}}\,dx = \lim_{M\to 1^+}\{arc\sec x\}_M^2 = arc\sec 2 -$

$\lim_{M\to 1^+}\{arc\sec M\} = \frac{\pi}{3} - 0 = \frac{\pi}{3}$.

Thus, $\int_1^2 \frac{1}{x\sqrt{x^2-1}}\,dx$ converges to $\frac{\pi}{3}$.

(15) $\int_0^\infty xe^{-x}\,dx = \lim_{M\to\infty} \int_0^\infty xe^{-x}\,dx = \lim_{M\to\infty}\left\{\frac{-x}{e^x} - \frac{1}{e^x}\right\}_0^M = \lim_{M\to\infty}\left\{\frac{-M}{e^M} - \frac{1}{e^M}\right\} - \{0-1\}$

$= \lim_{M\to\infty}\left\{\frac{-1}{e^M} - 0\right\} + 1 = 1$. Thus, $\int_0^\infty xe^{-x}\,dx$ converges to 1.

(16) $\int_1^\infty \frac{1}{1+x^2}\,dx =$

$\lim_{M\to\infty} \int_1^M \frac{1}{1+x^2}\,dx = \lim_{M\to\infty}\{\arctan x\}_1^M = \lim_{M\to\infty}\{\arctan M\} - \arctan 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.

Thus, $\int_1^\infty \frac{1}{1+x^2}\,dx$ converges to $\frac{\pi}{4}$.

(17) $\int_0^{\infty} \frac{x}{1+x^2}\, dx$ =

$\lim_{M\to\infty} \int_0^{M} \frac{x}{1+x^2}\, dx = \lim_{M\to\infty} \left\{ \frac{1}{2} \ln\left|1+x^2\right| \right\}_0^M = \lim_{M\to\infty} \left\{ \frac{1}{2} \ln\left|1+M^2\right| \right\} - 0 = \infty - 0 = \infty$. Thus,

$\int_0^{\infty} \frac{x}{1+x^2}\, dx$ diverges.

(18) $\int_e^{\infty} \frac{1}{x\ln x}\, dx = \lim_{M\to\infty} \int_e^{M} \frac{1}{x\ln x}\, dx = \lim_{M\to\infty} \{\ln\left|\ln|x|\right|\}_e^M = \lim_{M\to\infty} \{\ln\left|\ln|M|\right|\} - \ln\left|\ln|e|\right| = \infty - 0 = \infty$.

Thus, $\int_e^{\infty} \frac{1}{x\ln x}\, dx$ diverges.

(19) $\int_0^{\infty} \frac{\arctan x}{1+x^2}\, dx = \lim_{M\to\infty} \int_0^{M} \frac{\arctan x}{1+x^2}\, dx = \lim_{M\to\infty} \left\{ \frac{1}{2} (\arctan x)^2 \right\}_0^M$

$$= \lim_{M\to\infty} \left\{ \frac{1}{2} (\arctan M)^2 \right\} - 0 = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{8}.$$

Thus, $\int_0^{\infty} \frac{\arctan x}{1+x^2}\, dx$ converges to $\frac{\pi^2}{8}$.

(20) $\int_1^{e} \frac{1}{x\ln x}\, dx = \lim_{M\to 1^+} \int_M^{e} \frac{1}{x\ln x}\, dx = \lim_{M\to 1^+} \{\ln\left|\ln|x|\right|\}_M^e = \ln\left|\ln|e|\right| - \lim_{M\to 1^+} \{\ln\left|\ln|M|\right|\} =$

$0 - (-\infty) = \infty$.

Thus, $\int_1^{e} \frac{1}{x\ln x}\, dx$ diverges.

(21) $\int_1^{e} \frac{1}{x(\ln x)^2}\, dx = \lim_{M\to 1^+} \int_M^{e} \frac{1}{x(\ln x)^2}\, dx = \lim_{M\to 1^+} \left\{ \frac{-1}{\ln x} \right\}_M^e = \frac{-1}{1} - \lim_{M\to 1^+} \left\{ \frac{-1}{\ln M} \right\} =$

$-1 + \infty = \infty$. Thus, $\int_1^{e} \frac{1}{x(\ln x)^2}\, dx$ diverges.

3.8 Gamma and Beta functions:

In this section, we discuss the Gamma and beta functions. These functions arise in the solution of physical problems and are also of great importance in various branches of mathematical analysis.

3.8.1 Euler's integrals:

Definition of Beta function:

The definite integral $\int_0^1 [x^{m-1}](1-x)^{n-1}$dx, for m >0, n >0 is known as the beta function and is denoted by B(m, n). Beta function is also called the Eulerian integral of the first kind. Thus, B (m, n) = $\int_0^1 [x^{m-1}](1-x)^{n-1}$dx, for m>0,n>0 ……….……………(1)

Definition of Gamma function:

The definite integral $\int_0^\infty [e^{-x}](x)^{n-1}$dx, for, n>0 ………………………… (2) is known as the gamma function and is denoted by $\Gamma_{(n)}$. Gamma function is also called the Eulerian integral of the second kind.

3.8.2 Properties of Gamma function:

1. To show that $\Gamma_{(1)}$=1

Proof: By the definition of Gamma function

$\Gamma_{(n)}=\int_0^\infty [e^{-x}](x)^{n-1}$dx, ……………….(1)

From (1) $\Gamma_{(1)} = \int_0^1 [e^{-x}](x)^{1-1}$dx, $=\int_0^\infty [e^{-x}]$ dx =1

2. To show that $\Gamma_{(n+1)}$= n$\Gamma_{(n)}$, n.>0

Proof: By the definition of Gamma function $\Gamma_{(n+1)}=\int_0^\infty [e^{-x}](x)^{n+1-1}$dx

$=\int_0^\infty [e^{-x}](x)^{n}$dx

$=(x^n)e^{-x}$-$\int_0^\infty [e^{-x}](nx)^{n-1}$dx, on integrating by parts

$\Gamma_{(n+1)}= \lim_{n\to\infty} \frac{x^n}{e^x}$ +0 +n $\int_0^\infty [e^{-x}](nx)^{n-1}$dx ……… (1) Now we have

$$\lim_{n\to\infty} \frac{x^n}{e^x} = \lim_{n\to\infty} \frac{x^n}{1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\ldots\frac{x^n}{n!}+\frac{x^{n-1}}{(n+1)}+\cdots} = \lim_{n\to\infty} \frac{1}{\frac{1}{x^n}+\frac{1}{1!x^{n-1}}+\cdots+\cdots\frac{1}{n!}+\frac{x}{(n+1)}+\cdots} = 0$$

Also by definition $\Gamma_{(n)}=\int_0^\infty [e^{-x}](x)^{n-1}dx$

Using the above facts (1) reduces to $\Gamma_{(n+1)}= n\Gamma_{(n)}$

3. If n is non-negative integer, then $\Gamma_{(n+1)}= n!$

   Proof: we know that for n>0, we have (from property (2))

   $\Gamma_{(n+1)}= n\Gamma_{(n)}= n\Gamma_{(n-1+1)} =n(n-1)\Gamma_{(n-1)}$ by property (2) again

   $= n\ (n-1)\ (n-2)\ldots\ldots\Gamma_{(1)}$ (by repeated use of property 2 and the fact that n is positive integer)

   $=n!$ as $\Gamma_{(1)}=1$

Extension of definition Gamma function $\Gamma_{(n)}$ for n>0

When n>0, we known that $\Gamma_{(n+1)}= n\Gamma_{(n)}$

So that $\Gamma_{(n)} = \frac{\Gamma(n+1)}{n}$ ……………… (1)

Let -1 < n < 0. Then -1<n implies $n+1>0$ so that $\Gamma_{(n+1)}$ is well defined by definition and so R.H.S of (1) is well defined. Thus $\Gamma_{(n)}$ is defined for -1 < n <0 by (1). Similarly, $\Gamma_{(n)}$ is given by (1) for -2 < n <-1. -3 < n < -2 and so on. Thus (1) defined $\Gamma_{(n)}$ for all values of n except n=0 ,-1,-2,-3,…….

3.8.3 Property:

To show that $\Gamma_{(n)} =\infty$, if n is zero or a negative integer.

Proof: putting n = 0 in (1), we get $\Gamma_{(0)} = \frac{\Gamma(1)}{0}$ implies $\Gamma_{(0)} = \infty$ ………..……….(3)

Again, putting n = -1 in (1), we get $\Gamma_{(-1)} = \frac{\Gamma(0)}{-1}$ i$\Rightarrow \Gamma_{(-1)} = \infty$ by (2) ……………. (4)

Next putting n = -2 in (1), and using (3) we get $\Gamma_{(-2)} = \frac{\Gamma(-1)}{-2} \Rightarrow \Gamma_{(-2)} = \infty$, and so on.

Thus, we find that $\Gamma_{(n)}$ is ∞ if n is zero or negative integer.

3.8.4 Theorem: To show that $\Gamma_{(\frac{1}{2})} = \sqrt{\pi}$

Proof: From definition of gamma function $\Gamma_{(n)}=\int_0^\infty [e^{-t}](t)^{n-1}dt$ ……… (1)

Replacing n by $\frac{1}{2}$ in (1), we have

$\Gamma_{(\frac{1}{2})}=\int_0^\infty [e^{-t}](t)^{-\frac{1}{2}}dt$

$= 2\int_0^\infty [e^{-u^2}]$du ………..(2)

[Pitting t=u^2 so that dt=2u du]

$\therefore \Gamma_{(\frac{1}{2})} = 2\int_0^\infty [e^{-x^2}]$dx and $\Gamma_{(\frac{1}{2})}=2\int_0^\infty [e^{-y^2}]$dy …………(3)

[Limits remaining the same, we can write x or y as the variable in the integral of (2)]. Multiplying the corresponding sides of two equations of (3), we get

$(\Gamma_{(\frac{1}{2})})^2 = 2\int_0^\infty [e^{-x^2}]$dx $=2\int_0^\infty [e^{-y^2}]$dy

$= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}$dydx$=4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r d\theta$ dr (on changing the variable to polar co-ordinates (r, θ) where x= r cosθ, y= r sinθ, so that $x^2 + y^2 = r^2$ and

dx dy = rdθ dr. The area of integration is the positive quadrant of xy-plane)

$\therefore (\Gamma_{(\frac{1}{2})})^2 = 2\int_0^{\frac{\pi}{2}} [\int_0^\infty 2e^{-r^2} r dr] d\theta = 2\int_0^{\frac{\pi}{2}} [\int_0^\infty e^{-v)} dv] d\theta$, putting $r^2 = v$ so that 2rdr=dv. Hence $(\Gamma_{(\frac{1}{2})})^2 = 2\int_0^{\frac{\pi}{2}} [-e^{-v}]\, d\theta = 2\int_0^{\frac{\pi}{2}} d\theta = 2[\theta]_0^{\frac{\pi}{2}} = \pi$

Thus $(\Gamma_{(\frac{1}{2})})^2 = \pi$ -----------(4)

Remark, from (3) and (4) $2\int_0^\infty [e^{-x^2}]$dx=$\sqrt{\pi}$

3.8.5 Transformation of gamma function:

Form 1: To show that $\Gamma_{(n)} = \frac{1}{n}\int_0^\infty \left[e^{-x^{\frac{1}{n}}}\right]$dx, n>0

Proof: By definition $\Gamma_{(n)} = \int_0^\infty [e^{-x}](x)^{n-1}$dx, n>0 ……….(1)

[Putting t=x^n so that $n(x)^{n-1}$dx= dt] then (1) gives $\Gamma_{(n)} = \frac{1}{n}\int_0^\infty \left[e^{-x^{\frac{1}{n}}}\right]$dt

(or) $\Gamma_{(n)} = \frac{1}{n}\int_0^\infty \left[e^{-x^{\frac{1}{n}}}\right]$dx …….(2)

Particular Case n=$\frac{1}{2}$ in (ii), Then $\Gamma_{(\frac{1}{2})} = 2\int_0^\infty [e^{-x^2}]$dx ………………..(3)

Form 2: Show that $\frac{\Gamma(n)}{k^n} = \frac{1}{n}\int_0^\infty [e^{-kx}]\, x^{n-1}$dx, n>0, k>0

Proof: By definition $\Gamma_{(n)} = \int_0^\infty [e^{-x}](x)^{n-1}$dx, n>0 ……….(1)

[Putting kt=x so that dx= kdt] then (1) gives

$\Gamma_{(n)} = \frac{1}{n}\int_0^\infty [e^{-it}]\, k^{n-1} t^{n-1}$kdt (or)

$\Gamma_{(n)} = k^n \int_0^\infty [e^{-kx}]\, x^{n-1}$dx (or)

$\int_0^\infty [e^{-kx}]\, x^{n-1}$dx $= \frac{\Gamma(n)}{k^n}$

Form 3: To Show that $\Gamma_{(n)} = \int_0^1 (log\frac{1}{x})^{n-1}$dx, n>0

Proof: By definition $\Gamma_{(n)} = \int_0^\infty [e^{-x}](x)^{n-1}$dx, n>0 ……….(1)

[Putting t=e^{-x} so that -e^{-x}dx= dt] then (1) gives

$\Gamma_{(n)} - \int_0^1 (log\frac{1}{x})^{n-1}$dt as t=$e^{-x}$ $\frac{1}{t}$=$e^{x} \Rightarrow x = log\frac{1}{t}$

$\Gamma_{(n)} = \int_0^1 (log\frac{1}{x})^{n-1}$dx, n>0

Form 4: To show that $\Gamma_{(n)} = 2\int_0^\infty [e^{-x^2}]\, x^{2n-1}$dx, n>0

Proof: By definition $\Gamma_{(n)} = \int_0^\infty [e^{-x}](x)^{n-1}$dx, n>0 ……….(1)

[Putting x=t^2 so that dx= 2tdt] then (1) gives

$\Gamma_{(n)} = \int_0^\infty \left[e^{-x^2} (t^2)^{n-1}\right] 2t$dt or $\Gamma_{(n)} = 2\int_0^\infty \left[e^{-x^2} (t)^{2n-1}\right] dx$

3.8.6 Solved examples based on Gamma function:

Problem 1:

i) $\int_0^\infty [e^{-x}(x)^4]$dx ii) $\int_0^\infty [e^{-2x}(x)^6]$dx

Solution: $\int_0^\infty [e^{-x}(x)^4]$dx $= \int_0^\infty e^{-x}(x)^{5-1}$dx $= \Gamma_{(5)} = 4! = 24$ by definition of gamma function

iii) I=$\int_0^\infty [e^{-2x}(x)^6]$dx put 2x = t so that dx=$\frac{1}{2}$dt, then we have) I=$\int_0^\infty \left[e^{-t} (\frac{t}{2})^6 \frac{1}{2}\right]$dt

$= \frac{1}{2^7}\int_0^\infty [e^{-t}(t)^{7-1}]dt$

$= \frac{1}{2^7}\Gamma_{(7)}$, by definition of gamma distribution

$= \frac{1}{2^7}6! = \frac{45}{8}$

Problem 2

i)$\Gamma_{(-\frac{1}{2})}$ ii) $\Gamma_{(-\frac{3}{2})}$ iii) $\Gamma_{(-\frac{5}{2})}$

Solution: We know that $\Gamma_{(n)} = \frac{\Gamma(n+1)}{n}$ ……….(1)

Part (i), Putting n=$-\frac{1}{2}$ in (1). $\Gamma_{(-\frac{1}{2})} = \frac{\Gamma_{(\frac{1}{2})}}{(-\frac{1}{2})}$

$=\frac{\sqrt{\pi}}{(-\frac{1}{2})}$

$=-2\sqrt{\pi}$, as $\Gamma_{(\frac{1}{2})} = \sqrt{\pi}$

Part (ii), Putting n=$-\frac{3}{2}$ in (1) we have . $\Gamma_{(-\frac{3}{2})} = \frac{\Gamma_{(\frac{1}{2})}}{(-\frac{3}{2})} = \frac{2}{3}\Gamma_{(-\frac{1}{2})}$

$=\frac{2}{3}(-2\sqrt{\pi})$

$=\frac{4}{3}\sqrt{\pi}$as $\Gamma_{(\frac{1}{2})}$

$=\sqrt{\pi}$

Part (iii), Putting n=$-\frac{5}{2}$ in (1) we have . $\Gamma_{(-\frac{5}{2})} = \frac{\Gamma_{(-\frac{1}{2})}}{(-\frac{5}{2})} = \frac{2}{5}\frac{4}{3}\sqrt{\pi}$ using part (ii)

Example 1: If n is a positive integer, prove that $2^n\Gamma_{(1+\frac{1}{2})}$ =1.3.5…(2n+1)$\sqrt{\pi}$

Using the formula n $\Gamma_{(n)} = \Gamma_{(n+1)}$, n>0 ………………(1)

$\Gamma_{(n+\frac{1}{2})} = \Gamma\left(n-\frac{1}{2}+1\right)$

$=(n-\frac{1}{2})\Gamma\left(n-\frac{1}{2}\right)$

$=(n-\frac{1}{2})\,\Gamma\left(n-\frac{3}{2}+1\right)$ using (1)

$=(n-\frac{1}{2})\,(n-\frac{3}{2})\,\Gamma\left(n-\frac{3}{2}\right)$

$=\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\Gamma\left(\frac{2n-3}{2}\right)$

$=\left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma_{(\frac{1}{2})}$

{by repeated application of (1) and noting that (2n-1)(2n-3)….. are all odd}

$\therefore \Gamma_{(n+\frac{1}{2})} = \frac{(2n-1)(2n-3).....5.3.1}{2^n}\sqrt{\pi}$ as $\Gamma_{(\frac{1}{2})}$

$=\sqrt{\pi}$

$2^n\Gamma_{\left(n+\frac{1}{2}\right)}=(2n-1)(2n-3)\ldots..5.3.1\sqrt{\pi}$

Example 2: If n is a positive integer and m >-1, Prove that

$\int_0^1 (x^m)[(logx)^2]\,dx = \left(\frac{n!(-1)^n}{(m+1)^{n+1}}\right)$

Solution: Let I= $\int_0^1 (x^m)[(logx)^2]\,dx$ Put $logx$ =-t so that x=e^{-t} and dx=-$e^{-t}dt$

I= $\int_0^1 (e^{-t})^m)[(-t)^n](-e^{-t})\,dt$,[∴ logU=-∞ and log 1=0]

=$(-1)^n \int_0^\infty (e^{-t}(^{m+1})[(t)^{(n+1)-1}]$

=$(-1)^n \frac{\Gamma(n+1)}{(m+1)^{n+1}}$, provided m+1 >0i.e m>-1

=$(-1)^n \frac{n!}{(m+1)^{n+1}}$ [$\Gamma_{(n+1)} = n!$, n being the integer]

Example 3: With certain limitations on the value of a, b, m and n prove that

$$\int_0^\infty \int_0^\infty e^{-(ax^2+bx^2)}\,[x^{2m-1}]\,y^{2m-1}dxdy = \frac{\Gamma_{(m)}\Gamma_{(n)}}{4b^n a^m}$$

Solution: Let *I*=$\int_0^\infty \int_0^\infty e^{-(ax^2+bx^2)}\,[x^{2m-1}]\,y^{2m-1}dxdy = \frac{\Gamma(m)\Gamma(n)}{4b^n a^m}$ ….(1)

(or)

I=$\int_0^\infty e^{-(ax^2)}[x^{2m-1}]dx \int_0^\infty e^{-(ax^2)}\,y^{2m-1}dy = I_1 * I_2$ ………..….(2)

Where I_1 =$\int_0^\infty e^{-(ax^2)}[x^{2m-1}]dx$ ………………….……………(3)

I_2 =$\int_0^\infty e^{-(ax^2)}\,y^{2m-1}dy$ ………………..……..…………..(4)

Put ax^2=t, i.e x= $\left(\frac{t}{a}\right)^{\frac{1}{2}}$ so that dx=$\frac{dt}{2\sqrt{at}}$. Then (3) becomes

I_1 =$\int_0^\infty e^{-(t)}\left[\left(\frac{t}{a}\right)^{\frac{(2m-1)}{2}}\right]\frac{dt}{2\sqrt{at}}$

= $\frac{1}{2a^m}\int_0^\infty e^{-(t)}[(t)^{(m-1)}]dt$

=$\frac{\Gamma(m)}{2a^m}$, by definition of gamma function, taking m>0, a>0 similarly I_2=$\frac{\Gamma(n)}{2a^n}$, n>0, b>0

∴ From (1) and (2), we have I =$I_1 * I_2$=$\frac{\Gamma(m)}{4a^m}\frac{\Gamma(n)}{a^n}$

UNIT - IV

MATRICES

4.Introduction:

Matrix is a rectangular array of real numbers. We will use the *double subscript* notation, that is the element from matrix A in the i^{th} row and the j^{th} column is denoted by a_{ij} .

- The dimensions of a matrix are given by *rows* $\times$ *columns* (order $m \times n$).
- A matrix is a square if it is $n \times n$, then we say it has order n. The main diagonal of a square matrix is all the elements, a_{ij} , with $i = j$.
- Matrices relate to systems of equations - we can write the system of equations without the variables, addition signs, and equal signs. So, if the system of equations is:
- $a_{11}x + a_{12}y + a_{13}z = b_1$
- $a_{21}x + a_{22}y + a_{23}z = b_2$
- $a_{31}x + a_{32}y + a_{33}z = b_3$

Then the augmented matrix is $\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$ and the coefficient matrix is $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Note that any time a term is missing from the system of equations we must put a zero in its place in the matrix.

1. A rectangular array of m×n numbers arranged in the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called an **m×n *matrix***.

e.g. $\begin{bmatrix} 2 & 3 & 4 \\ 1 & -8 & 5 \end{bmatrix}$ is a 2×3 matrix.

e.g. $\begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ is a 3×1 matrix.

2. If a matrix has m *rows* and n *columns*, it is said to be ***order*** **m×n**.

e.g. $\begin{bmatrix} 2 & 0 & 3 & 6 \\ 3 & 4 & 7 & 0 \\ 1 & 9 & 2 & 5 \end{bmatrix}$ is a matrix of order 3×4.

e.g. $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 5 \\ -1 & 3 & 0 \end{bmatrix}$ is a matrix of order 3.

3. $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ is called a ***row matrix*** **or** ***row vector***.

4. $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is called a ***column matrix*** **or** ***column vector***.

e.g. $\begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}$ is a column vector of order 3×1.

e.g. $\begin{bmatrix} -2 & -3 & -4 \end{bmatrix}$ is a row vector of order 1×3.

5. If all elements are real, the matrix is called **a real matrix**.

6. $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ is called a ***square matrix*** of order n and $a_{11},\ a_{22},\ \ldots,\ a_{nn}$ is called *the principal diagonal.*

(e.g) $\begin{bmatrix} 3 & 9 \\ 0 & -2 \end{bmatrix}$ is a square matrix of order 2.

7. Notation: $\left[a_{ij}\right]_{m\times n}$, $\left(a_{ij}\right)_{m\times n}$, A , ...

4.1 Some Special Matrix:

If all the elements are zero, the matrix is called a *zero matrix* or null matrix, denoted by $O_{m\times n}$.

e.g. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 2×2 zero matrix, and denoted by O_2. Let $A = \left[a_{ij}\right]_{n\times n}$ be a square matrix.

(i) If $a_{ij} = 0$ for all i, j, then A is called a **zero matrix**.

ii) If $a_{ij} = 0$ for all i<j, then A is called a ***lower triangular matrix***.

(iii) If $a_{ij} = 0$ for all i>j, then A is called an ***upper triangular matrix***.

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & & \vdots \\ \vdots & & & & 0 \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ 0 & 0 & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

i.e. Lower triangular matrix Upper triangular matrix

e.g.1 $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 4 \end{bmatrix}$ is a lower triangular matrix.

e.g.2 $\begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix}$ is an upper triangular matrix.

4.2 Diagonal matrix.

Let $A = [a_{ij}]_{n\times n}$ be a square matrix. If $a_{ij} = 0$ for all $i \neq j$, then A is called a ***diagonal matrix***.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix.

If A is a diagonal matrix and $a_{11} = a_{22} = \cdots = a_{nn} = 1$, then A is called an ***identity matrix* or a *unit matrix*, denoted by** I_n**.**

e.g. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4.3 Arithmetic's of Matrices:

Two matrices A and B are equal if they are of the *same order* and their *corresponding elements are equal*.

i.e. $[a_{ij}]_{m\times n} = [b_{ij}]_{m\times n} \quad \Leftrightarrow \quad a_{ij} = b_{ij}$ for all i, j .

e.g. $\begin{bmatrix} a & 2 \\ 4 & b \end{bmatrix} = \begin{bmatrix} -1 & c \\ d & 1 \end{bmatrix} \Leftrightarrow \quad a=-1,\ b=1,\ c=2,\ d=4.$

$$\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix} \ \&$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}$$

Let $A = [a_{ij}]_{m\times n}$ & $B = [b_{ij}]_{m\times n}$.

Define $A+B$ as the matrix $C = [c_{ij}]_{m\times n}$ of the same order such that $c_{ij} = a_{ij} + b_{ij}$ for all i=1, 2..., m and j=1, 2..., n.

e.g. $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -4 & 3 \\ 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 2 \\ 3 & o1 & 9 \end{bmatrix}$

1. $\begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}$ is not defined.

2. $\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} + 5$ is not defined.

Let $A = [a_{ij}]_{m\times n}$.

Then $-A = [-a_{ij}]_{m\times n}$

and A-B=A+(-B)

e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix}$ &

$$B = \begin{bmatrix} 2 & 4 & 0 \\ 3 & -1 & 1 \end{bmatrix}.$$

Find -A and A-B.

-A = $\begin{bmatrix} -1 & -2 & -3 \\ 1 & 0 & -2 \end{bmatrix}$

A-B = $\begin{bmatrix} -1 & -2 & 3 \\ -4 & 1 & 1 \end{bmatrix}$

4.4 Properties of Matrix Addition:

Let A, B, C be matrices of the same order and O be the zero matrix of the same order. Then

(a) A + B=B + A

(b) (A+B) + C=A+(B+C)

(c) A+ (-A) = (-A) +A=O

(d) A + O=O + A

4.5 Scalar Multiplication:

Let $A = \left[a_{ij}\right]_{m\times n}$, k is scalar. Then kA is the matrix $C = \left[c_{ij}\right]_{m\times n}$ defined by

$c_{ij} = ka_{ij}$, $\forall$i, j i.e. $kA = \left[ka_{ij}\right]_{m\times n}$

e.g. If $A = \begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}$,

Then -2A=$\begin{bmatrix} -6 & 2 \\ 10 & -12 \end{bmatrix}$

(1) -A = (-1) A

(2) A – B=A + (-1) B

4.5.1 Properties of Scalar Multiplication:

Let A, B be matrices of the same order and h, k be two scalars. Then

(a) k(A+B) =kA + kB

(b) (k + h) A=kA + hA

(c) (hk)A=h(kA)=k(hA)

4.5.2 Definition: Transpose Matrix $A = \left[a_{ij}\right]_{m\times n}$. The *transpose* of A, denoted by A^T, or A', is defined by

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nm} \end{bmatrix}_{n\times m}$$

e.g. $A = \begin{bmatrix} 3 & -2 \\ -5 & 6 \end{bmatrix}$,

the $A^T = \begin{bmatrix} 3 & 5 \\ 2 & 6 \end{bmatrix}$

e.g. $A = \begin{bmatrix} 3 & 0 & -2 \\ 4 & -6 & 1 \end{bmatrix}$,

then $A^T = \begin{bmatrix} 3 & 4 \\ 0 & 6 \\ 2 & 1 \end{bmatrix}$ e.g

$A = [5]$, then

$A^T = [5]$

4.5.3 Properties of Transpose:

Let A, B be two m×n matrices and k be a scalar, then

(a) $(A^T)^T = A$

(b) $(A+B)^T = A^T + (B)^T$

(c) $(kA)^T = A^T$k

4.6 Symmetric matrix:

A square matrix A is called a *symmetric matrix* if

$$A^T = A.A^T$$

i.e. A is symmetric matrix

$\Leftrightarrow A^T = A \Leftrightarrow a_{ij} = a_{ji} \quad \forall$i, j

e.g. $\begin{bmatrix} 1 & 3 & -1 \\ 3 & -3 & 0 \\ -1 & 0 & 6 \end{bmatrix}$ is a symmetric matrix.

e.g. $\begin{bmatrix} 1 & 3 & -1 \\ 0 & -3 & 0 \\ -1 & 3 & 6 \end{bmatrix}$ is not a symmetric matrix?

4.6.1 Skew-Symmetric:

Definition: Square matrix A is called a *skew-symmetric matrix* if

$$A^T = -A.$$

i.e. A is skew-symmetric matrix

$\Leftrightarrow A^T = -A \Leftrightarrow a_{ij} = -a_{ji} \quad \forall$i, j

e.g. $A = \begin{bmatrix} 0 & 3 & -1 \\ -3 & 0 & 5 \\ 1 & -5 & 0 \end{bmatrix}$ is a skew-symmetric matrix.

4.7 Conjugate of a matrix:

The matrix obtained from any given matrix A, on replacing its elements by the corresponding conjugate numbers is called the conjugate of A and denoted by $\overline{A}$

Example $A = \begin{bmatrix} 1+i & 2-3i & -1 \\ 2+2i & -i & 3-2i \\ -2i & 5+8i & 0 \end{bmatrix}$

and $\overline{A} = \begin{bmatrix} 1-i & 2+3i & -1 \\ 2-2i & +i & 3+2i \\ +2i & 5-8i & 0 \end{bmatrix}$

4.8 Matrix Multiplication:

Let $A = [a_{ik}]_{m\times n}$ & $B = [b_{kj}]_{n\times p}$. Then the product AB is defined as the m×p matrix

$C = [c_{ij}]_{m\times p}$ where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

i.e. $$AB = \left[\sum_{k=1}^{n} a_{ik}b_{kj}\right]_{m\times p}$$

e.g. Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}_{2\times 3}$. Find AB and BA.

e.g. Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}_{2\times 2}$, Find AB. Is BA well defined?

In general, AB≠BA. i.e. matrix multiplication is not commutative.

4.8.1 Properties of Matrix Multiplication:

(a) (AB)C = A(BC)

(b) A(B+C) = AB+AC

(c) (A+B) C = AC+BC

(d) AO = OA = O

(e) IA = AI = A

(f) k(AB) = (kA)B = A(kB)

(g) $(AB)^T = B^T A^T$.

(1) Since AB ≠BA;

Hence, A(B+C) $\neq$ (B+C) A & A(kB) $\neq$ (kB)A.

(2) $A^2 + kA = A(A + kI) = (A + kI)A$.

(3) $AB - AC = O \Rightarrow A(B - C) = O$

$\not\Rightarrow A = O$ or $B - C = O$

e.g. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, Then

$$AB - AC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ But A $\neq$ O and B $\neq$ C, so

$AB - AC = O \not\Rightarrow A = O$ or $B = C$.

4.9 Determinants:

Definition: Let $A = [a_{ij}]$ be a square matrix of order n. The determinant of A, det A or |A| is defined as follows:

(a) If n=2, $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

(b) If n=3, $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ (or)

$$\det A = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

e.g. Evaluate (a) $\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix}$

(b) $\det\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 1 & -2 & -1 \end{bmatrix}$

e.g. If $\begin{vmatrix} 3 & 2 & x \\ 8 & x & 1 \\ 3 & -2 & 0 \end{vmatrix} = 0$, find the value(s) of x.

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

or $$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

or . . . .

By using $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Exercise:

Evaluate (a) $\begin{vmatrix} 3 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{vmatrix}$

(b) $\begin{vmatrix} 0 & 2 & 0 \\ 8 & -2 & 1 \\ 3 & 2 & 3 \end{vmatrix}$

4.9.1 Properties of Determinants:

(1) $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{i.e.} \quad \det(A^T) = \det A.$$

(2) $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

(3) $$\begin{vmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & 0 & c_3 \end{vmatrix} = 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 0 \end{vmatrix}$$

(4) $$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(5) If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$, then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(6) $$\begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}$$

(7) $$\begin{vmatrix} pa_1 & b_1 & c_1 \\ pa_2 & b_2 & c_2 \\ pa_3 & b_3 & c_3 \end{vmatrix} = p\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ pa_2 & pb_2 & pc_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} pa_1 & pb_1 & pc_1 \\ pa_2 & pb_2 & pc_2 \\ pa_3 & pb_3 & pc_3 \end{vmatrix} = p^3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(1) $$\begin{pmatrix} pa_1 & pb_1 & pc_1 \\ pa_2 & pb_2 & pc_2 \\ pa_3 & pb_3 & pc_3 \end{pmatrix} = p\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

(2) If the order of A is n, then $\det(\lambda A) = \lambda^n \det(A)$

(8) $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \lambda b_1 & b_1 & c_1 \\ a_2 + \lambda b_2 & b_2 & c_2 \\ a_3 + \lambda b_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \overset{\alpha C_2 + \beta C_3 + C_1}{=\!=\!=\!=\!=\!=\!=\!=} \begin{vmatrix} x_1 + \alpha y_1 + \beta z_1 & y_1 & z_1 \\ x_2 + \alpha y_2 + \beta z_2 & y_2 & z_2 \\ x_3 + \alpha y_3 + \beta z_3 & y_3 & z_3 \end{vmatrix}$$

Exercise:

(1) Evaluate (a) $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix}$,

(b) $\begin{vmatrix} 5 & 3 & 7 \\ 3 & 7 & 5 \\ 7 & 2 & 6 \end{vmatrix}$

(2) Evaluate $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

(3) Factorize the determinant

$$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

(4) Factorize each of the following :

(a) $$\begin{vmatrix} a^3 & b^3 & c^3 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

(b) $$\begin{vmatrix} 2a^3 & 2b^3 & 2c^3 \\ a^2 & b^2 & c^2 \\ 1-a^3 & 1-b^3 & 1-c^3 \end{vmatrix}$$

4.9.2 Multiplication of Determinants:

$$Let\ |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

$$|B| = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

Then $$|A||B| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix}$$

4.9.3 Minors and Cofactors:

Definition: Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then A_{ij} , the *cofactor* of a_{ij} , is defined by

$$A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\ A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},\ \ldots,\ A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Since $$|A| = -a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= +a_{21}A_{21} - a_{22}A_{22} + a_{23}A_{23}$$

4.9.4 Theorem: (a) $$a_{i1}A_{j1} + a_{i2}A_{j2} + a_{i3}A_{j3} = \begin{cases} \det A & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(b) $$a_{1i}A_{1j} + a_{2i}A_{2j} + a_{3i}A_{3j} = \begin{cases} \det A & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

e.g. $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = \det A$, $\quad a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$, etc.

e.g. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and c_{ij} be the cofactor of a_{ij}, where $1 \leq i, j \leq 3$.

(a) Prove that $A\begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} = |\mathrm{A}|\, I$

(b) Hence, deduce that $\begin{vmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{vmatrix} = (|\mathrm{A}|)^2$

4.10 Inverse of Square Matrix By Determinants:

Definition: The *cofactor matrix* of A is defined as $cofA = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$.

Def. The *adjoint matrix* of A is defined as

$$adjA = (cofA)^T$$
$$= \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}.$$

e.g. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find adjA.

e.g. (a) Let $A = \begin{pmatrix} 1 & 1 & -3 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, find adjA.

(b) Let $B = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & -1 \\ 5 & 1 & -1 \end{pmatrix}$, find adjB.

e.g. Given that $A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & -1 \\ 5 & 1 & -1 \end{pmatrix}$, find A^{-1}.

e.g. Suppose that the matrix $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-singular, find A^{-1}.

e.g. Given that $A=\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$, find A^{-1}.

4.10.1 Theorem: A square matrix A is non-singular if $|A| \neq 0$.

e.g. Show that $A=\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ is non-singular.

e.g. Let $A=\begin{pmatrix} x+1 & 2 & x-1 \\ x-1 & 2 & -1 \\ 5 & 7 & -x \end{pmatrix}$, where $x \in R$.

(a) Find the value(s) of x such that A is non-singular.

(b) If x=3, find A^{-1}.

A is singular (non-invertible) if A^{-1} does not exist. Then

A square matrix A is singular if $|A| = 0$.

4.10.2 Properties of Inverse matrix:

Let A, B be two non-singular matrices of the same order and λ be a scalar.

(1) $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$

(2) $(A^{-1})^{-1} = A$

(3) $(A^{T})^{-1} = \left(A^{-1}\right)^{T}$

(4) $(A^{n})^{-1} = \left(A^{-1}\right)^{n}$ for any positive integer n.

(5) $(AB)^{-1} = B^{-1}A^{-1}$

(6) The inverse of a matrix is unique.

(7) $\det(A^{-1}) = \frac{1}{\det A}$

(8) If $M=\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, Then $M^{-1}=\begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{pmatrix}$.

(9) If $M=\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$,

then $M^n = \begin{pmatrix} a^n & 0 & 0 \\ 0 & b^n & 0 \\ 0 & 0 & c^n \end{pmatrix}$ where n ≠0.

e.g. Let $A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix}$,

$B = \begin{pmatrix} 1 & -3 & -1 \\ 0 & 13 & 4 \\ 0 & -33 & -10 \end{pmatrix}$

and $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

(a) Find A^{-1} and M^5.

(b) Show that $ABA^{-1} = M$.

(c) Hence, evaluate B^5.

e.g. Let $A = \begin{pmatrix} 3 & 8 \\ 1 & 5 \end{pmatrix}$

& $P = \begin{pmatrix} 2 & -4 \\ 1 & 1 \end{pmatrix}$.

(a) Find $P^{-1}AP$.

(b) Find A^n, where n is a positive integer

e.g. (a) Show that if A is a 3x3 matrix such that $A^t = -A$, then $|A|$=0.

(b) Given that $B = \begin{pmatrix} 1 & -2 & 74 \\ 2 & 1 & -67 \\ -74 & 67 & 1 \end{pmatrix}$,

Use (a), or otherwise, to show $\det(I - B) = 0$.

Hence deduce that $\det(I - B^4) = 0$.

$$x^3 - 38x^2 + 361x - 900 = 0.$$

4.10.3 Inverse of a Square Matrix:

If a, b, c are real numbers such that ab=c and b is non-zero, then

$a = \dfrac{c}{b} = cb^{-1}$ and b^{-1} is usually called the multiplicative inverse of b.

If B, C are matrices, then $\frac{C}{B}$ is undefined.

<u>4.10.4 Definition</u>: A square matrix A of order n is said to be non-singular or invertible if and only if there exists a square matrix B such that AB = BA = I. The matrix B is called the multiplicative inverse of A, denoted by A^{-1} i.e. $AA^{-1} = A^{-1}A = I$.

<u>4.10.5 Definition:</u> If a square matrix A has an inverse, A is said to be *non-singular* or *invertible*. Otherwise, it is called *singular* or *non-invertible*.

e.g. $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ And $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ are both non-singulars.

i.e. A is non-singular if A^{-1} exists.

<u>4.10.6 Theorem:</u> The inverse of a non-singular matrix is unique.

(1) $I^{-1} = I$,so, I is always non-singular.

(2) OA = O $\neq$ I, so O is always singular.

(3) Since AB = I imply BA = I.

Hence proof of either AB = I or BA = I is enough to assert that B is the inverse of A.

e.g. Let $A = \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$.

(a) Show that $I - 6A + A^2 = O$.

(b) Show that A is non-singular and find the inverse of A.

(c) Find a matrix X such that $AX = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$.

Theorem: Let A, B be two non-singular matrices of the same order and λ be a scalar.

(a) $(A^{-1})^{-1} = A$.

(b) $A^T \sum$ is a non-singular and $(A^T)^{-1} = (A^{-1})^T$.

(c) A^n is a non-singular and $(A^n)^{-1} = (A^{-1})^n$.

(d) λA is a non-singular and $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$.

(e) AB is a non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.

Exercise

1. Given $A = \begin{pmatrix} 2 & -5 \\ 6 & 7 \end{pmatrix}$

$B = \begin{pmatrix} 0 & 6 \\ 2 & -3 \end{pmatrix}$

$C = \begin{pmatrix} 7 & 2 \\ 5 & 1 \end{pmatrix}$ find:

a) A + B

b) C – A

c) 3A

d) 4B + 2C

2. Given $A = \begin{pmatrix} 2 & 2 \\ 4 & 0 \\ 6 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 5 & 0 \\ 2 & 9 \end{pmatrix}$

and $C = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

a) Is AB defined? Calculate AB. Can you calculate BA?

b) Is BC defined? Calculate BC. Is CB defined? If so calculate CB.

c) Is it the case that BC = CB?

3. Find product matrices for the following:

a) $\begin{pmatrix} 2 & -1 & 4 \\ 3 & 0 & -7 \\ 5 & 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 2 & 0 \\ 3 & 5 \end{pmatrix}$

b) $\begin{pmatrix} -3 & 2 & 5 \\ 4 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

4. Given $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$ caculate:

a) AI b) IA c) BI d) IB

5. Given A and B as defined in question 4 find:

a) A'   b) B'

6. Given A = $\begin{pmatrix} 2 & 4 \\ 5 & -2 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & 0 \\ -1 & 5 \end{pmatrix}$ find   a) A^{-1} b) B^{-1}

4.11 Solving a System of linear Equations Using Matrices:

Solving a 2×2 system of linear equations by using the inverse matrix method

A system of linear equations can be solved by using our knowledge of inverse matrices.

The steps to follow are:

1.Express the linear system of equations as a matrix equation.

2.Determine the inverse of the coefficient matrix.

3.Multiply both sides of the matrix equation by the inverse matrix.

4. To multiply the matrices on the right side of the equation.

5.The inverse matrix must appear in front of the answer matrix. (the number of columns in the first matrix must equal the number of rows in the second matrix). The solution will appear as: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where c_1 *and* c_2 are the solutions.

Examples: Solve the following system of linear equations by using the inverse matrix method:

1. $\begin{Bmatrix} 2x + 9y = -1 \\ 4x + y = 15 \end{Bmatrix}$

Solution: $\begin{pmatrix} 2 & 9 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 15 \end{pmatrix}$ This is the matrix equation that represents the system.

If $A = \begin{pmatrix} 2 & 9 \\ 4 & 1 \end{pmatrix}$ then $|A| = 2 - 36$

$|A| = -34$

$$A^{-1} = \begin{pmatrix} \frac{1}{-34} & \frac{-9}{-34} \\ \frac{-4}{-34} & \frac{2}{-34} \end{pmatrix} \quad A^{-1} = \begin{pmatrix} \frac{-1}{34} & \frac{9}{34} \\ \frac{4}{34} & \frac{-2}{34} \end{pmatrix}$$

This is the determinant and the inverse of the coefficient matrix.

$$\begin{pmatrix} \frac{-1}{34} & \frac{9}{34} \\ \frac{4}{34} & \frac{-2}{34} \end{pmatrix}\begin{pmatrix} 2 & 9 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-1}{34} & \frac{9}{34} \\ \frac{4}{34} & \frac{-2}{34} \end{pmatrix}\begin{pmatrix} -1 \\ 15 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-2}{34}+\frac{36}{34} & \frac{-9}{34}+\frac{9}{34} \\ \frac{8}{34}+\frac{-8}{34} & \frac{36}{34}+\frac{-2}{34} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{34}+\frac{135}{34} \\ \frac{-4}{34}+\frac{-30}{34} \end{pmatrix}$$

$$\begin{pmatrix} \frac{34}{34} & \frac{0}{34} \\ \frac{0}{34} & \frac{34}{34} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{136}{34} \\ \frac{-34}{34} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$ the common point or solution is (4, -1).

This is the result of multiplying the matrix equation by the inverse of the coefficient matrix.

2. $\begin{Bmatrix} 3x-6y=45 \\ 9x-5y=-8 \end{Bmatrix}$

Solution: $\begin{pmatrix} 3 & -6 \\ 9 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 45 \\ -8 \end{pmatrix}$

If $A = \begin{pmatrix} 3 & -6 \\ 9 & -5 \end{pmatrix}$ then $\begin{matrix} |A| = -15 - -54 \\ |A| = 39 \end{matrix}$

$$A^{-1} = \begin{pmatrix} \frac{-5}{39} & \frac{6}{39} \\ \frac{-9}{39} & \frac{3}{39} \end{pmatrix}$$

$$\begin{pmatrix} \frac{-5}{39} & \frac{6}{39} \\ \frac{-9}{39} & \frac{3}{39} \end{pmatrix}\begin{pmatrix} 3 & -6 \\ 9 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-5}{39} & \frac{6}{39} \\ \frac{-9}{39} & \frac{3}{39} \end{pmatrix}\begin{pmatrix} 45 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-15}{39}+\frac{54}{39} & \frac{30}{39}+\frac{-30}{39} \\ \frac{-27}{39}+\frac{27}{39} & \frac{54}{39}+\frac{-15}{39} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-225}{39}+\frac{-48}{39} \\ \frac{-405}{39}+\frac{-24}{39} \end{pmatrix}$$

$$\begin{pmatrix} \frac{39}{39} & \frac{0}{39} \\ \frac{0}{39} & \frac{39}{39} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-273}{39} \\ \frac{-429}{39} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 \\ -11 \end{pmatrix}$$ The common point or solution is (-7, -11).

In the next example, the products will be written over the common denominator instead of being written as two separate fractions.

3. $\begin{Bmatrix} 4x + y = -13 \\ -6x - 5y = 37 \end{Bmatrix}$

Solution: $\begin{pmatrix} 4 & 1 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -13 \\ 37 \end{pmatrix}$

If $A = \begin{pmatrix} 4 & 1 \\ -6 & -5 \end{pmatrix}$ then $\begin{aligned} |A| &= -20 - -6 \\ |A| &= -14 \end{aligned}$

$$A^{-1} = \begin{pmatrix} \frac{-5}{-14} & \frac{-1}{-14} \\ \frac{6}{-14} & \frac{4}{-14} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{5}{14} & \frac{1}{14} \\ \frac{-6}{14} & \frac{-4}{14} \end{pmatrix}$$

$$\begin{pmatrix} \frac{5}{14} & \frac{1}{14} \\ \frac{-6}{14} & \frac{-4}{14} \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -6 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{5}{14} & \frac{1}{14} \\ \frac{-6}{14} & \frac{-4}{14} \end{pmatrix}\begin{pmatrix} -13 \\ 37 \end{pmatrix}$$

$$\begin{pmatrix} \frac{20 + -6}{14} & \frac{5 + -5}{14} \\ \frac{-24 + 24}{14} & \frac{-6 + 20}{14} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-65 + 37}{14} \\ \frac{78 + -148}{14} \end{pmatrix}$$

$$\begin{pmatrix} \frac{14}{14} & \frac{0}{14} \\ \frac{0}{14} & \frac{14}{14} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-28}{14} \\ \frac{-70}{14} \end{pmatrix}$$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ The common point or solution is (-2, -5).

4. $\begin{Bmatrix} 3x - y = -11 \\ x + 2y = 8 \end{Bmatrix}$

Solution: $\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -11 \\ 8 \end{pmatrix}$

If $A=\begin{pmatrix}3 & -1\\ 1 & 2\end{pmatrix}$ then $|A| = 6 - -1$

$|A| = 7$

$$A^{-1} = \begin{pmatrix}\frac{2}{7} & \frac{1}{7}\\ \frac{-1}{7} & \frac{3}{7}\end{pmatrix}$$

$$\begin{pmatrix}\frac{2}{7} & \frac{1}{7}\\ \frac{-1}{7} & \frac{3}{7}\end{pmatrix}\begin{pmatrix}3 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\frac{2}{7} & \frac{1}{7}\\ \frac{-1}{7} & \frac{3}{7}\end{pmatrix}\begin{pmatrix}-11\\ 8\end{pmatrix}$$

$$\begin{pmatrix}\frac{6+1}{7} & \frac{-2+2}{7}\\ \frac{-3+3}{7} & \frac{1+6}{7}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\frac{-22+8}{7}\\ \frac{11+24}{7}\end{pmatrix}$$

$$\begin{pmatrix}\frac{7}{7} & \frac{0}{7}\\ \frac{0}{7} & \frac{7}{7}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\frac{-14}{7}\\ \frac{35}{7}\end{pmatrix} \quad \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-2\\ 5\end{pmatrix}$$

The common point or solution is (-2, 5)

Exercises: Solve the following systems of linear equations by using the inverse matrix method:

1. $\begin{Bmatrix}-5x+3y=21\\ -2x+7y=-21\end{Bmatrix}$ 2. $\begin{Bmatrix}2x+3y=48\\ 3x+2y=42\end{Bmatrix}$

3. $\begin{Bmatrix}2x-6y=3\\ 4x-3y=5\end{Bmatrix}$ 4. $\begin{Bmatrix}-x+y=1\\ -4x+2y=8\end{Bmatrix}$

Answers:

Solving systems of linear equations using the inverse matrix method

1. $\begin{Bmatrix}-5x+3y=21\\ -2x+7y=-21\end{Bmatrix}$ If $A=\begin{pmatrix}-5 & 3\\ -2 & 7\end{pmatrix}$ then

$|A| = -35 - -6$

$|A| = -29$

$$A^{-1}=\begin{pmatrix}\frac{7}{-29} & \frac{-3}{-29}\\ \frac{2}{-29} & \frac{-5}{-29}\end{pmatrix}\Rightarrow A^{-1}=\begin{pmatrix}\frac{-7}{29} & \frac{3}{29}\\ \frac{-2}{29} & \frac{5}{29}\end{pmatrix}$$

$$\begin{pmatrix}-5 & 3\\ -2 & 7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}21\\ -21\end{pmatrix}$$

$$\begin{pmatrix}\frac{-7}{29} & \frac{3}{29}\\ \frac{-2}{29} & \frac{5}{29}\end{pmatrix}\begin{pmatrix}-5 & 3\\ -2 & 7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\frac{-7}{29} & \frac{3}{29}\\ \frac{-2}{29} & \frac{5}{29}\end{pmatrix}\begin{pmatrix}21\\ -21\end{pmatrix}$$

$$\begin{pmatrix}\frac{35+-6}{29} & \frac{-21+21}{29}\\ \frac{10-10}{29} & \frac{-6+35}{29}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\frac{-147+-63}{29}\\ \frac{-42+-105}{29}\end{pmatrix}$$

$$\begin{pmatrix}\frac{29}{29} & \frac{0}{29}\\ \frac{0}{29} & \frac{29}{29}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\frac{-210}{29}\\ \frac{-147}{29}\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-7.24\\ -5.07\end{pmatrix}$$

2. $\begin{Bmatrix}2x+3y=48\\ 3x+2y=42\end{Bmatrix}$ If $A=\begin{pmatrix}2 & 3\\ 3 & 2\end{pmatrix}$

then $|A|=4-+9$ $|A|=-5$

$$A^{-1}=\begin{pmatrix}\frac{2}{-5} & \frac{-3}{-5}\\ \frac{-3}{-5} & \frac{2}{-5}\end{pmatrix}\Rightarrow A^{-1}=\begin{pmatrix}\frac{-2}{5} & \frac{3}{5}\\ \frac{3}{5} & \frac{-2}{5}\end{pmatrix}$$

$$\begin{pmatrix}2 & 3\\ 3 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}48\\ 42\end{pmatrix}$$

$$\begin{pmatrix}\frac{-2}{5} & \frac{3}{5}\\ \frac{3}{5} & \frac{-2}{5}\end{pmatrix}\begin{pmatrix}2 & 3\\ 3 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\frac{-2}{5} & \frac{3}{5}\\ \frac{3}{5} & \frac{-2}{5}\end{pmatrix}\begin{pmatrix}48\\ 42\end{pmatrix}$$

$$\begin{pmatrix} \frac{-4+9}{5} & \frac{-6+6}{5} \\ \frac{6+-6}{5} & \frac{9+-4}{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-96+126}{5} \\ \frac{144+-84}{5} \end{pmatrix}$$

$$\begin{pmatrix} \frac{5}{5} & \frac{0}{5} \\ \frac{0}{5} & \frac{5}{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{30}{5} \\ \frac{60}{5} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

3. $\begin{Bmatrix} 2x-6y=3 \\ 4x-3y=5 \end{Bmatrix}$

If $A = \begin{pmatrix} 2 & -6 \\ 4 & -3 \end{pmatrix}$

then $|A| = -6 - -24$,

$|A| = 18$

$$A^{-1} = \begin{pmatrix} \frac{-3}{18} & \frac{6}{18} \\ \frac{-4}{18} & \frac{2}{18} \end{pmatrix}$$

$$\begin{pmatrix} 2 & -6 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-3}{18} & \frac{6}{18} \\ \frac{-4}{18} & \frac{2}{18} \end{pmatrix}\begin{pmatrix} 2 & -6 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-3}{18} & \frac{6}{18} \\ \frac{-4}{18} & \frac{2}{18} \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-6+24}{18} & \frac{18+-18}{18} \\ \frac{-8+8}{18} & \frac{24+-6}{18} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-9+30}{18} \\ \frac{-12+10}{18} \end{pmatrix}$$

$$\begin{pmatrix} \frac{18}{18} & \frac{0}{18} \\ \frac{0}{18} & \frac{18}{18} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{21}{18} \\ \frac{-2}{18} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.1\overline{6} \\ -.1\overline{1} \end{pmatrix}$$

4. $\left\{\begin{matrix} -x+y=1 \\ -4x+2y=8 \end{matrix}\right\}$ If $A=\begin{pmatrix} -1 & 1 \\ -4 & 2 \end{pmatrix}$ then

$|A| = -2 - -4$ & $A^{-1}=\begin{pmatrix} \frac{2}{2} & \frac{-1}{2} \\ \frac{4}{2} & \frac{-1}{2} \end{pmatrix}$ $\begin{pmatrix} -1 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 8 \end{pmatrix}$

$$\begin{pmatrix} \frac{2}{2} & \frac{-1}{2} \\ \frac{4}{2} & \frac{-1}{2} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \frac{2}{2} & \frac{-1}{2} \\ \frac{4}{2} & \frac{-1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-2+4}{2} & \frac{2+-2}{2} \\ \frac{-4+4}{2} & \frac{4+-2}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \frac{2+-8}{2} \\ \frac{4+-8}{2} \end{pmatrix}$$

$$\begin{pmatrix} \frac{2}{2} & \frac{0}{2} \\ \frac{0}{2} & \frac{2}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \frac{-6}{2} \\ \frac{-4}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

4.12 Elementary Transformations of Matrices:

Elementary transformations of a matrix find a whole application in various mathematical problems. For example, they by in a basis of the known gauss method (method of exception of unknown values) for solution of linear equations

Elementary transformations of a matrix are:

1. Rearrangement of two rows (Columns)
2. Multiplication of all row (Column) elements of a matrix
3. Addition of two rows (Columns) of the matrix multiplied by the same number, not equal to zero.

Two matrices are called equivalent if one of them is maybe received from another after final number of elementary transformations. Generally equivalent matrixes are not equal, but have the same rank.

Calculations of determinants by means of Elementary transformations:

By means of Elementary transformations, it is easy to calculate a determinant of a matrix. For example, it is required to calculate a determinant of the matrix.

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & & a_{14} & \dots. & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{24} & \dots\dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & & a_{34} & \dots\dots & a_{3n} \\ .. & \vdots & \dots. & \ddots & \dots.. & \vdots & \dots.. \\ \dots & \dots. & \dots.. & \cdots & \dots.. & \dots.. & \dots\dots \\ a_{n1} & a_{n2} & a_{n3} & & a_{n4} & \dots\dots. & a_{nn} \end{vmatrix}$$

Where $a_{11} \neq 0$ then it is possible to bear multiplier a_{11} ,

$$\Delta = a_{11} \begin{vmatrix} 1 & a_{12} & a_{13} & & a_{14} & \dots. & a_{1n} \\ \frac{a_{21}}{a_{11}} & a_{22} & a_{23} & \cdots & a_{24} & \dots\dots & a_{2n} \\ \frac{a_{31}}{a_{11}} & a_{32} & a_{33} & & a_{34} & \dots\dots & a_{3n} \\ .. & \vdots & \dots. & \ddots & \dots.. & \vdots & \dots.. \\ \dots & \dots. & \dots.. & \cdots & \dots.. & \dots.. & \dots\dots \\ \frac{a_{n1}}{a_{11}} & a_{n2} & a_{n3} & & a_{n4} & \dots\dots. & a_{nn} \end{vmatrix}$$

now multiplying from elements if the j^{th} column ($j \geq 2$) appropriating elements of the column, multiplied on a_{1j}. We will receive the determinant

$$\Delta = a_{11} \begin{vmatrix} 1 & 0 & 0 & & 0 & \dots. & 0 \\ \frac{a_{21}}{a_{11}} & a_{22}^1 & a_{23}^1 & \cdots & a_{24}^1 & \dots\dots & a_{2n}^1 \\ \frac{a_{31}}{a_{11}} & a_{32}^1 & a_{33}^1 & & a_{34}^1 & \dots\dots & a_{3n}^1 \\ .. & \vdots & \dots. & \ddots & \dots.. & \vdots & \dots.. \\ \dots & \dots. & \dots.. & \cdots & \dots.. & \dots.. & \dots\dots \\ \frac{a_{n1}}{a_{11}} & a_{n2}^1 & a_{n3}^1 & & a_{n4}^1 & \dots\dots. & a_{nn}^1 \end{vmatrix}$$

which is equal to

$\Delta = a_{11}\Delta_{n-1}$ where

$$\Delta_{n-1} = \begin{bmatrix} a_{22}^1 & a_{23}^1 & \dots\dots\dots & a_{2n}^1 \\ a_{n2}^1 & a_{n3}^1 & \dots\dots\dots. & a_{nn}^1 \end{bmatrix}$$

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{11} \quad a_{1j}}{a_{11}} \quad (i,j = 2,3,4\dots\dots n)$$

Then we repeat the same actions for Δ_{n-k} and, if all elements $a_{ij}^{(i-1)} \neq 0$ (j = 2,3,4…….n), then we will receive finally

$$\Delta = a_{11}a_{22}^{(1)}\dots\dots\dots a_{nn}^{(n-1)}$$

If for any intermediate determinant Δ_{n-k} its left upper element $a_{k-1,k-1}^{(k)}$=0, it is necessary to rearrange rows or column in Δ_{n-k} so that a new left upper element will not be equal to zero. If $\Delta \neq 0$ it always can be made. Thus it is ncecssary to consider that the sign on a determinant changes on what element a_{pp} is the main one(that is when the matrix is transformed so that a_{pp}=1). Then the sign on an appropriating determinant is equal to $(-1)^{p+q}$

Example:1 by mean of Elementary transformations result the matrix

A=$\begin{pmatrix} 1 & 2 & -3 \\ 4 & -1 & 3 \\ 6 & 10 & 5 \end{pmatrix}$ to a tringle type

Solution: Frist we will multiply the first row of the matrix by 4, and the second by (-1) and add the first row to the second

A'=$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 9 & -15 \\ 6 & 10 & 5 \end{pmatrix}$, now we will multiply the first row of the matrix by 4, and the third by (-1) and add the first row to the third A''=$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 9 & -15 \\ 6 & 2 & -23 \end{pmatrix}$, finally we will multiply the second row of the matrix by 2, and the third by (-1) and add the second row to the third A'''=$\begin{pmatrix} 1 & 2 & -3 \\ 0 & 9 & -15 \\ 0 & 0 & 177 \end{pmatrix}$. As a result the upper triangular matrix A''' is received.

<u>4.13 Elementray Matrices :</u>

A matrix obtain form a unit matrix, by subjecting it to any of the elementry transformations is called an elementray matrices.

<u>4.13.1 Symbols for Elementray Matrices:</u>

I .E_{ij} will also denote the matrix obtain by interchaging the i^{th} and j^{th} columns, for, as may easily be seen, the matrices obtained by interchaging the i^{th} and j^{th} rows or the i^{th} and j^{th} columns of a unit matrix are the same.

II. (a) E_i (c) will denote the matrix obtained by multiplying the i^{th}row of the unit matrix c.

It the first from I in only one position, viz the (i,i)th

I		E_i (c)	
(i,i)	(i,j)	(i,i)	(i,j)
1	0	c	0

$b. E_i$ (c) will also denote the matrix obtained by multiplying the $i^{th} column$ by c.

III. (a) E_{iJ} (k) will denote the matrix, obtained by adding to the elements $of the\ i^{th}$row of the unit matrix, the products by k odf the corresponding elements of the j^{th}

It differs from I in only one place, viz the (i,j)th

I		E_{iJ} (k)	
(i,i)	(i,j)	(i,i)	(i,j)
1	0	c	0

$b)\ E'_{ij}$(k) which is the transpose of E_{iJ} (k) will denote the matrix obtained by adding to the elements of the $i^{th} column,$ the products by k of the corresponding elements of the j^{th}

4.13.2 Determinants of Elementary Matrices:

It is easy to see that $|E_{iJ}|$ = -1, $|E_i\ (c)|$ =c $\neq$0

$|E_{iJ}\ (k)|$ = $|\ E'_{ij}(k)|$ =1 So that every elementary matrix is non –singular. This fact also shows the basis of our insisting that, c, must not be zero

4.13.3 Definition Of Row Rank And Column Rank:

The dimension of the row space of A is called the row rank of A and the dimension of the column space of A is called the column rank of A.

Since the basis of the row space of A is

$$\{[1\ \ 0\ \ 2\ \ 0\ \ 1],[0\ \ 1\ \ 1\ \ 0\ \ 1],[0\ \ 0\ \ 0\ \ 1\ \ -1]\},$$

the dimension of the row space is 3 and the row rank of A is 3. Similarly,

$$\left\{\begin{bmatrix}1\\3\\2\\-1\end{bmatrix},\begin{bmatrix}-2\\2\\3\\2\end{bmatrix},\begin{bmatrix}3\\1\\2\\4\end{bmatrix}\right\}$$

is the basis of the column space of A. Thus, the dimension of the column space is 3 and the column rank of A is 3.

- **Important Result:**

The row rank and column rank of the $m \times n$ matrix A are equal.

4.13.4 Definition of the Rank of a Matrix:

Since the row rank and the column rank of a $m \times n$ matrix A are equal, we only refer to the rank of A and write $rank(A)$.

- **Important Result:**

If A is a $m \times n$ matrix, then

$$rank(A) + nullity(A)$$
$$= \text{the dimension of the column space} + \text{the dimension of the null space}$$
$$= n$$

$$A = \begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix} \text{ and } n = 5.$$

Since

$$\left\{\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}\right\}$$

is a basis of column space and thus $rank(A) = 3$. The solutions of $Ax = 0$ are

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = s_1, x_5 = s_2,\ s_1, s_2 \in R.$$

Thus, the solution space (the null space) is \ $s_1\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}+s_2\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} \Leftrightarrow span\left\{\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$.

Then, $\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$ are the basis of the null space. and $nullity(A)=2$.

Therefore, rank(A) + nullity(A) = 3+2 =5= n

- **<u>Important Result:</u>**

Let A be $n\times n$ matrix. A is non-singular if and only if $rank(A)=n$

rank(A) = n $\Leftrightarrow$ A is non-singular det(A) $\neq$0

$$rank(A)<n \quad \Leftrightarrow \quad Ax=0 \text{ has a nontrivial solution.}$$

4.13.5 Reduction to Normal From:

Theorem: Every no zero matrix of rank r, can by a sequence of elementary transformations, be reduced to the form

$$\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$$

I_r being the unit matrix. The from obtained here is the normal form. Let A be a given non-zero matrix. Since A$\neq$ 0, it has at least one non-zero element.

Let a_{ij} =k $\neq$ 0

By interchanging the i^{th} row with the first row and the j^{th} column with the first column, we obtain a matrix B such that b_{11} =k $\neq$ 0

Dividing the element of the first row by k, we obtain a matrix C such that a_{11}=1.

Subtracting from the elements of the first column by c_{ij}we obtain a matrix D such that d_{ij} =0

As in the preceding step, subtracting from each of the other column and rows, suitable multiple of the first column and first row respectively, we obtain a matrix E such that each of the column in its first row and column, excepting the one in the (1,1)$((1,1)^{th})$ place, is zero. Then E is the form

$$\begin{pmatrix} I_r & O \\ O & A_1 \end{pmatrix}$$

If, now, $A_1 \neq$O, we can deal with it as we did with A_1, without affecting the first row and the first column.

Thus proceeding, we shall obtain a diagonal matrix of the given form.

Since elementary transformations do not alter the rank, the finally obtained diagonal matrix, whose rank is r, must have, r and only r non-zero elements.

Note:

If a matrix B is obtained from a matrix A is an elementary transformation, we write A ~B.

4.13.6 Equivalence of Matrices:

Definition: Let A ϵ $M_{m\,xn}$(F) and B ϵ $M_{m\,xn}$(F). A is said to be equivalent to B, if there exists two non-singular matrices, P ,Q whose elements are member of f such that

A= PBQ

The following theree properties of this relation are fundamental

I Reflexivity : Every matrix, A, is equivalent to itself, for we have A $= |A|$ so that P = I, Q= I

II Symmetry: If A, is equivalent to B over F, then B is also equivalent to A over F, for

A= PBQ ⇒ P=$P^{-1}AQ^{-1}$where $P^{-1}, Q^{-1}$are non -singular matrices over F

III.Transitivity: If A, is equivalent to B over Fand B is equivalent to C over F, then A is also equivalent to C over F for A= PBQ B= LCM

⇒ A=$PLCMQ = (PL)C(MQ)$ where PL , MQ , being the products of non-singular matrices are non -singular matrices over F

Because of the these properties of non singular matrices over F, are relation "equivalance of matrices over F, is reflectivie, symmetiric and transitive."

4.13.7 Criterian for Equivalance:

Theorem: 1

The nx n matrices over a field F are equivalance over F, if and only if they have the same rank.

Let A, B be equivalance over F. There exists non-singular matrices P,Q, over F such that A= PBQ

As multiplication with a non-singular matrix does not after the rank , the rank of A and B are the same.

Let now A,B have the same rank, r . If A and B are both equivalance over F to the matrix

$$\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$$

so that, because of the symmetic and the transitivity of the equivalance relation, the matrices A, B are equivalent over F.

4.13.8 Canonical matrices for equivalance over a Field, class Partitions:

Because of the these fundamental properties of reflexivity, symmetiric and transitivity, the relation of equivalance of matrices over F divides the set of all m x n matrices over F into a system of mutually exclusive classes such that

i) each member of the set belongs to same class.

ii) two members of the same class are equivalent.

iii) No two members of two different classes are equivalent.

Again by the theorem in (i) above, we see that each class is uniquely characterize by the rank of any of its members so that the rank is invariant for members of a same class.

As the rank of an m x n matricx can assume any value between 0 and

(m , n) say = k, we see that the number of different classes obtain by the equivalence relation, in question, is k+1.

0, $\begin{pmatrix} I_1 & O \\ O & O \end{pmatrix} \begin{pmatrix} I_2 & O \\ O & O \end{pmatrix} \ldots\ldots \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$

Their ranks respectively are 0,1,2, 3…….k

Each of these (k+1) matrices is a representative of one of the (k+1) classes referred to above in the sense that

i. Each of the (k+1) classes contains one matrix of the above set, and
ii. Each member of the set belongs to some class. These (k+1) member are said is to be the canonical matrices for the set of (m x n) matrices over a field F, with respect to the relation of equivalence of matrices over F. Every (m x n) matrix over F is equivalence to one and only one canonical matrix.

UNIT – V

5.1 Characteristic roots and vectors:

Let $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$ and let x denote a 2x1 matrix.

(a) Find the two real values λ_1 and λ_2 of λ with $\lambda_1 > \lambda_2$

Such that the matrix equation

(*) $Ax = \lambda x$ has non-zero solutions.

(b) Let x_1 and x_2 be non-zero solutions of (*) corresponding to λ_1 and λ_2 respectively. Show that if $x_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$ and $x_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$ then the matrix $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ is non-singular.

(c) Using (a) and (b), show that $AX = X\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

and hence $A^n = X\begin{pmatrix} \lambda_1^{\ n} & 0 \\ 0 & \lambda_2^{\ n} \end{pmatrix}X^{-1}$ where n is a positive integer.

Example 1:

Find the Eigenvalues and Eigen vectors of $A = \begin{pmatrix} -1 & -26 \\ 1 & -3 \end{pmatrix}$. For the eigenvalues one has $A - \lambda I = \begin{pmatrix} -1-\lambda & -26 \\ 1 & -3-\lambda \end{pmatrix}$

$$0 = |A - \lambda I|$$

$$= \begin{vmatrix} -1-\lambda & -26 \\ 1 & -3-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(-3-\lambda) - (1)(-26)$$

$$= \lambda^2 + 4\lambda + 3 + 26$$

$$= \lambda^2 + 4\lambda + 29$$

So the eigenvalues are

$$\lambda = \frac{-4 \pm \sqrt{(-4)^2 - (4)(1)(29)}}{(2)(1)}$$

$$= \frac{-4 \pm \sqrt{100}}{2}$$

$$= -2 \pm 5i$$

So $\lambda_1 = -2 + 5i$ &

$\lambda_2 = -2 - 5i$

This example illustrates a general feature of complex eigenvalues of matrices that have real entries, i.e. they occur in complex conjugate pairs. One reason is that the characteristic equation $|A - \lambda I| = 0$ is a polynomial equation in λ with real coefficients and for such equations the roots occur in complex conjugate pairs. We shall see another reason below.

For the eigenvectors for $\lambda_1 = -2 + 5i$ one has

$$A - \lambda_1 I = A - (-2 + 5i)I$$

$$= \begin{pmatrix} 1 - 5i & -26 \\ 1 & -1 - 5i \end{pmatrix}$$

So an eigenvector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfies

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (A - \lambda I)v = \begin{pmatrix} 1 - 5i & -26 \\ 1 & -1 - 5i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} (1 - 5i)x - 26y \\ x + (-1 - 5i)y \end{pmatrix}$$

So $(1 - 5i)\,x - 26y = 0$

$x + (-1 - 5i)y = 0$

If one multiplies the second equation by $1 - 5i$ one obtains the first. So, any solution to the second equation is also a solution to the first. So it suffices to solve the second equation whose solution is $x = (1 + 5i)y$. So, an eigenvector v for $\lambda_1 = -2 + 5i$ has the form

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (1 + 5i)y \\ y \end{pmatrix}$$

$$= y\begin{pmatrix} 1 + 5i \\ 1 \end{pmatrix}$$

So any multiple of the vector $v_1 = \begin{pmatrix} 1 + 5i \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda_1 = -2 + 5i$.

For $\lambda_2 = -2 - 5i$ all the previous computation that we did for $\lambda_1 = -2 + 5i$ remain the same except we replace i by $-i$. So, it is not hard to see that any multiple of the vector $\begin{pmatrix} 1 - 5i \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda_2 = -2 - 5i$.

This example illustrates a general feature of the eigenvectors for complex eigenvalues, namely the eigenvector for complex conjugate eigenvalues have complex conjugate components. It was not hard to see why this was true in the above example, and the same argument can be used in general. However, there is a slightly different argument that is useful in other similar situations.

If $z = x + yi$ is a complex number with real and imaginary parts x and y then the complex conjugate of z is $\overline{z} = x - yi$.

For example, $\overline{3 - 2i} = 3 + 2i$. The operation of taking complex conjugate has several simple algebraic properties. Some of these are

(1) $\overline{z + w} = \overline{z} + \overline{w}$

$\overline{z - w} = \overline{z} - \overline{w}$

(2) $\overline{zw} = \overline{z}\,\overline{w}$

$\left[\left(\frac{\overline{z}}{w}\right)\right] = \overline{z} / \overline{w}$

The operation of taking complex conjugates can be extended to vectors and matrices. If $v = \begin{pmatrix} z_1 \\ z_2 \\ \cdot \\ z_n \end{pmatrix}$ is a vector with complex components, then its complex conjugate is $\overline{v} = \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \\ \cdot \\ \overline{z_n} \end{pmatrix}$. If $\begin{pmatrix} a_{11}a_{12}\cdots a_{1n} \\ a_{21}a_{22}\cdots a_{2n} \\ \cdots \\ a_{m1}a_{m2}\cdots a_{mn} \end{pmatrix}$ is a matrix with complex components then its complex conjugate is $\overline{A} =$

$$\begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} & \cdots & \overline{a_{1n}} \\ \overline{a_{21}} & \overline{a_{22}} & \cdots & \overline{a_{2n}} \\ \cdots & & & \\ \overline{a_{m1}} & \overline{a_{m2}} & \cdots & \overline{a_{mn1}} \end{pmatrix}.$$

Example 2:

If $v = \begin{pmatrix} 2 - 3i \\ 5 + 4i \end{pmatrix}$ then $\overline{v} = \begin{pmatrix} 2 + 3i \\ 5 - 4i \end{pmatrix}$.

If $A = \begin{pmatrix} 2 - 3i & 7 + i \\ 5 + 4i & 6 - 8i \end{pmatrix}$ then $\overline{A} = \begin{pmatrix} 2 + 3i & 7 - i \\ 5 - 4i & 6 + 8i \end{pmatrix}$.

The algebraic properties (1) and (2) of complex conjugates for numbers extends to complex conjugates of vectors and matrices, e.g. if α is a complex number, u and v are vectors and A and B are matrices then

$$\overline{u+v} = \overline{u} + \overline{v} \qquad\qquad \overline{A+B} = \overline{A} + \overline{B}$$

$$(3) \qquad \overline{\alpha u} = \overline{\alpha}\,\overline{u} \qquad \overline{\alpha A} = \overline{\alpha}\,\overline{A} \qquad \overline{Au} = \overline{A}\,\overline{u} \qquad \overline{AB} = \overline{A}\,\overline{B}$$

The following proposition shows that complex eigenvalues of matrices with real entries occur in conjugate pairs.

5.2 Proposition :

Suppose A is a matrix with real entries and λ is an eigenvalue of A with eigenvector v. Then $\overline{\lambda}$ is also an eigenvalue of A and $\overline{v}$ is an eigenvector for $\overline{\lambda}$.

Proof: One has $Av = \lambda v$. Taking complex conjugates of both sides gives $\overline{Av} = \overline{\lambda v}$. Using (3) gives $\overline{A}\,\overline{v} = \overline{\lambda}\,\overline{v}$. Which proves the proposition?

Problem 1:

Consider the mapping that takes a point $v = \begin{pmatrix} x \\ y \end{pmatrix}$ and rotates it by an angle $\theta = \pi/4$ to the new point $w = \begin{pmatrix} r \\ s \end{pmatrix}$. We know that $w = Rv$ where $R = R_{\pi/4}$ is the matrix for a rotation by $\pi/4$. In general, the matrix R_θ for a rotation by θ is given by $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. In the case $\theta = \pi/4$ one has $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$. So, $R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. The eigenvalues of R cannot be real since no non-zero vector v is mapped on to the line through itself when it is rotated by $\pi/4$. To find the eigenvalues of R we proceed as usual.

$$R - \lambda I = \begin{pmatrix} \frac{1}{\sqrt{2}} - \lambda & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - \lambda \end{pmatrix}$$

$$0 \quad = \quad det(R - \lambda I) \quad = \quad \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - \lambda \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{2}} - \lambda\right)^2 + \frac{1}{2}$$

So $$\left(\frac{1}{\sqrt{2}} - \lambda\right)^2 \quad = \quad -\frac{1}{2} \Rightarrow \lambda - \frac{1}{\sqrt{2}}$$

$$= \pm\frac{1}{\sqrt{2}}$$

So, the eigenvalues are

$$\lambda_1 \quad = \quad \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

and $$\lambda_2 \quad = \quad -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

For the eigenvectors for $\lambda_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ one has

$$A - \lambda_1 I \; = \; A - (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)I$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$$

So, an eigenvector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfies

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad = \quad (A - \lambda I)v \quad = \quad \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -ix - y \\ x - iy \end{pmatrix}$$

So $$-ix - y \quad = \quad 0$$

$$x - iy \quad = \quad 0$$

If one multiplies the first equation by i one obtains the second. So any solution to the first equation is also a solution to the second. So it suffices to solve the first equation whose solution is $y = -iy$. So an eigenvector v for $\lambda_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ has the form $v = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -ix \end{pmatrix} =$

$x\begin{pmatrix} 1 \\ -i \end{pmatrix}$. So any multiple of the vector $v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector for $\lambda_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.

Taking complex conjugates one sees that $v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector for $\lambda_2 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

Problem 2: Show that the eigenvalues of $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ are $\lambda_1 = \cos\theta + (\sin\theta)\, i$ and $\lambda_2 = \cos\theta - (\sin\theta)\, i$ and the corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$.

Problem 3: Show that the eigenvalues of $A = \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}$ are $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$ and the corresponding eigenvectors are $v_1 = \begin{pmatrix} 2 \\ -1-i \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ -1+i \end{pmatrix}$

Problem 4:

Find the eigenvalues and eigenvectors of the matrix:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

A) First, we start by finding the eigenvalues, using the equation derived above:

$$|\mathbf{A} - \lambda\mathbf{I}| = \left| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}.$$

If you like, just consider this step as, “subtract λ from each diagonal element of the matrix in the question”.

Next, we derive a formula for the determinant, which must equal zero:

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 1 \times 1 = \lambda^2 - 2\lambda + 3 = 0.$$

We now need to find the roots of this quadratic equation in λ.

In this case the quadratic factories straightforwardly to:

$$\lambda^2 - 2\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0.$$

The solutions to this equation are $\lambda_1 = 1 \,\&\, \lambda_2 = 3$. These are the eigenvalues of the matrix $\mathbf{A}$.

We will now solve for an eigenvector corresponding to each eigenvalue in turn. First, we will solve for $\lambda = \lambda_1 = 1$:

To find the eigenvector we substitute a general vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ into the defining equation:

$$\mathbf{Ax} = \lambda\mathbf{x},$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1 \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

By multiplying out both sides of this equation, we form a set of simultaneous equations:

$$\begin{pmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ or}$$

$$2x_1 + x_2 = x_1,$$
$$x_1 + 2x_2 = x_2.$$

$$x_1 + x_2 = 0,$$
$$x_1 + x_2 = 0,$$

Where we have taken everything over to the LHS. It should be immediately clear that we have a problem as it would appear that these equations are not solvable! However, as we have already mentioned, the eigenvectors are not unique: we would not expect to be able to solve these equations for one value of x_1 and one value of x_2. In fact, all these equations let us do is specify a relationship between x_1 and x_2, in this case:

$$x_1 + x_2 = 0, \text{or, } x_2 = -x_1,$$

So, our eigenvector is produced by substituting this relationship into the general vector $\mathbf{x}$:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}.$$

This is a valid answer to the question; however, it is common practice to put 1 in place of x_1 and give the answer: $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

We follow the same procedure again for the second eigenvalue, $\lambda = \lambda_2 = 3$. First, we write

$$\mathbf{Ax} = \lambda\mathbf{x},$$

out the defining equation: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3 \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$

and multiply out to find a set of simultaneous equations:

$$2x_1 + x_2 = 3x_1,$$
$$x_1 + 2x_2 = 3x_2.$$

Taking everything over to the LHS we find:

$$-x_1 + x_2 = 0,$$
$$x_1 - x_2 = 0.$$

This time both equations can be made to be the same by multiplying one of them by minus one. This is used as a check: one equation should always be a simple multiple of the other; if they are not and can be solved uniquely then you have made a mistake.

Once again, we can find a relationship between x_1 and x_2, in this case $x_1 = x_2$, and form our general eigenvector: $\mathbf{x} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$.

As before, set $x_1 = 1$ to give: $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore our full solution is: $\begin{matrix} \lambda_1 = 1, & \mathbf{x}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}; \\ \lambda_2 = 3, & \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{matrix}$

Problem 5:

You will often be asked to find *normalized* eigenvectors. A normalized eigenvector is an eigenvector of length one. They are computed in the same way but at the end we divide by the length of the vector found. To illustrate, let's find the normalized eigenvectors and eigenvalues of the matrix:

$$\mathbf{A} = \begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix}.$$

A) First, we start by finding the eigenvalues using the eigenvalues equation:

$$|\mathbf{A} - \lambda \mathbf{I}| = \left| \begin{pmatrix} 5-\lambda & -2 \\ 7 & -4-\lambda \end{pmatrix} \right| = \mathbf{0}.$$

Computing the determinant, we find:

$(5-\lambda)(-4-\lambda) + 2\times 7 = 0,$ And multiplying out: $\lambda^2 - \lambda - 6 = 0.$

This quadratic can be factorized into $(\lambda-3)(\lambda+2)=0$, giving roots $\lambda_1 = -2$ & $\lambda_2 = 3$.

To find the eigenvector corresponding to $\lambda = \lambda_1 = -2$ we must solve:

$$\mathbf{Ax} = \lambda \mathbf{x},$$
$$\begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2 \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

When we compute this matrix multiplication we obtain the two equations:

$$5x_1 - 2x_2 = -2x_1,$$
$$7x_1 - 4x_2 = -2x_2.$$

Moving everything to the LHS we once again find that the two equations are identical:

$$7x_1 - 2x_2 = 0,$$
$$7x_1 - 2x_2 = 0,$$

And we can form the relationship $x_2 = \frac{7}{2}x_1$ and the eigenvector in this case is thus: $\mathbf{x} = \begin{pmatrix} x_1 \\ \frac{7}{2}x_1 \end{pmatrix}$.

In previous questions, we have set $x_1 = 1$, but we were free to choose any number. In this case things are made simpler by electing to use $x_1 = 2$ as this gets rid of the fraction, giving: $\mathbf{x} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$.

This is not the bottom line answer to this question as we were asked for normalized eigenvectors. The easiest way to normalize the eigenvector is to divide by its length, the length of this vector is:

$|\mathbf{x}| = \sqrt{2^2 + 7^2} = \sqrt{53}$. Therefore, the normalized eigenvector is: $\hat{\mathbf{x}} = \frac{1}{\sqrt{53}}\begin{pmatrix} 2 \\ 7 \end{pmatrix}$,

The chevron above the vector's name denotes it as normalised. It's a good idea to confirm that this vector does have length one:

$$|\hat{\mathbf{x}}| = \sqrt{\left(\frac{2}{\sqrt{53}}\right)^2 + \left(\frac{7}{\sqrt{53}}\right)^2} = \sqrt{\frac{4}{53} + \frac{49}{53}} = \sqrt{\frac{53}{53}} = 1.$$

We must now repeat the procedure for the eigenvalue $\lambda = \lambda_2 = 3$. We find the simultaneous equations are:

$$2x_1 - 2x_2 = 0,$$
$$7x_1 - 7x_2 = 0,$$

and note that they differ by a constant ratio. We find the relation between the components, $x_1 = x_2$, and hence the general eigenvector:

$\mathbf{x} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}$, and choose the simplest option $x_1 = 1$ giving: $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

This vector has length $\sqrt{1+1}=\sqrt{2}$, so the normalised eigenvector is: $\hat{\mathbf{x}}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$.

Therefore, the solution to the problem is:
$$\begin{aligned}\lambda_1&=-2, \quad \hat{\mathbf{x}}_1=\frac{1}{\sqrt{53}}\begin{pmatrix}2\\7\end{pmatrix};\\ \lambda_2&=3, \quad \hat{\mathbf{x}}_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.\end{aligned}$$

Problem 6:

Sometimes you will find complex values of λ; this will happen when dealing with a rotation matrix such as:

$$\mathbf{A}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

Which represents a rotation though $90°$. In this example, we will compute the eigenvalues and eigenvectors of this matrix.

A) First start with the eigenvalue formula:

$$|\mathbf{A}-\lambda\mathbf{I}|=\left|\begin{pmatrix}-\lambda & -1\\ 1 & -\lambda\end{pmatrix}\right|=\mathbf{0}.$$

Computing the determinant, we find: $\lambda^2+1=0$,

Which has complex roots $\lambda=\pm i$. This will lead to complex-valued eigenvectors, although there is otherwise no change to the normal procedure.

For $\lambda_1=i$ we find the defining equation to be:
$$\begin{gathered}\mathbf{Ax}=\lambda\mathbf{x},\\ \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=i\times\begin{pmatrix}x_1\\x_2\end{pmatrix}.\end{gathered}$$

Multiplying this out to give a set of simultaneous equations we find:

$$\begin{aligned}-x_2&=ix_1,\\ x_1&=ix_2.\end{aligned}$$

We can apply our check by observing that these two equations can be made the same by multiplying either one of them by i. This leads to the eigenvector: $\mathbf{x}=\begin{pmatrix}i\\1\end{pmatrix}$. Repeating this procedure for $\lambda=\lambda_2=-i$, we find: $\mathbf{x}=\begin{pmatrix}-i\\1\end{pmatrix}$. Therefore our full solution is:

$$\lambda_1 = i, \quad \mathbf{x}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix};$$

$$\lambda_2 = -i, \quad \mathbf{x}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

5.4 DEFINITION OF EIGENVALUES AND EIGENVECTORS OF A SQUARE MATRIX

If $[A]$ is a $n \times n$ matrix, then $[X] \neq \vec{0}$ is an eigenvector of $[A]$ if

$$[A][X] = \lambda[X]$$

Where λ is a scalar and $[X] \neq 0$. The scalar λ is called the eigenvalue of $[A]$ and $[X]$ is called the eigenvector corresponding to the eigenvalue λ.

Eigenvalues of a square matrix:

To find the eigenvalues of a $n \times n$ matrix $[A]$, we have

$$[A][X] = \lambda[X]$$

$$[A][X] - \lambda[X] = 0$$

$$[A][X] - \lambda[I][X] = 0$$

$$([A] - [\lambda][I])[X] = 0$$

Now for the above set of equations to have a nonzero solution,

$$\det([A] - \lambda[I]) = 0$$

This left-hand side can be expanded to give a polynomial in λ and solving the above equation would give us values of the eigenvalues. The above equation is called the characteristic equation of $[A]$.

For a $[A]$ $n \times n$ matrix, the characteristic polynomial of A is of degree n as follows

$\det([A] - \lambda[I]) = 0$, giving

$$\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + - - + c_n = 0$$

Hence, this polynomial has n roots.

Problem:7

Find the eigenvalues of the physical problem of the matrix

$$[A] = \begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix}$$

Solution:

$$[A]-\lambda[I]=\begin{bmatrix} 3-\lambda & -1.5 \\ -0.75 & 0.75-\lambda \end{bmatrix}$$

$$\det([A]-\lambda[I])=(3-\lambda)(0.75-\lambda)-(-0.75)(-1.5)=0$$

$$2.25-0.75\lambda-3\lambda+\lambda^2-1.125=0$$

$$\lambda^2-3.75\lambda+1.125=0$$

$$\lambda=\frac{-(-3.75)\pm\sqrt{(-3.75)^2-4(1)(1.125)}}{2(1)}$$

$$=\frac{3.75\pm3.092}{2}$$

$$=3.421, 0.3288$$

So, the eigenvalues are 3.421 and 0.3288.

Problem :8

Find the eigenvectors of

$$A=\begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix}$$

Solution: The eigenvalues have already been found in Example 1 as

$$\lambda_1=3.421, \lambda_2=0.3288$$

Let $[X]=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, be the eigenvector corresponding to

$\lambda_1=3.421$

Hence

$$([A]-\lambda_1[I])[X]=0$$

$$\left\{\begin{bmatrix} 3 & -1.5 \\ -0.75 & 0.75 \end{bmatrix}-3.421\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0$$

$$\begin{bmatrix} -0.421 & -1.5 \\ -0.75 & -2.671 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $x_1=s$ then

$$-0.421s-1.5x_2=0$$
$$x_2=-0.2808s$$

The eigenvector corresponding to $\lambda_1=3.421$ then is

$$[X]=\begin{bmatrix} s \\ -0.2808s \end{bmatrix}$$

$$= s\begin{bmatrix} 1 \\ -0.2808 \end{bmatrix}$$

The eigenvector corresponding to

$\lambda_1 = 3.421$ is

$$\begin{bmatrix} 1 \\ -0.2808 \end{bmatrix}$$

Similarly, the eigenvector corresponding to

$\lambda_2 = 0.3288$ is

$$\begin{bmatrix} 1 \\ 1.781 \end{bmatrix}$$

Problem 9:

Find the eigenvalues and eigenvectors of

Solution: The characteristic equation is given by

$$\det([A]-\lambda[I])=0$$

$$\det\begin{bmatrix} 1.5-\lambda & 0 & 1 \\ -0.5 & 0.5-\lambda & -0.5 \\ -0.5 & 0 & -\lambda \end{bmatrix}=0$$

$$(1.5-\lambda)[(0.5-\lambda)(-\lambda)-(-0.5)(0)]+(1)[(-0.5)(0)-(-0.5)(0.5-\lambda)]=0$$

$$-\lambda^3+2\lambda^2-1.25\lambda+0.25=0$$

To find the roots of the characteristic polynomial equation

$$-\lambda^3+2\lambda^2-1.25\lambda+0.25=0$$

We find that the first root by observation is $\lambda=1$

As substitution of $\lambda=1$ gives

$$(-1)^3+2(1)^2-1.25(1)+0.25=0$$

$0=0$ So $(\lambda-1)$ is a factor of

$-\lambda^3+2\lambda^2-1.25\lambda+0.25$.

To find the other factors of the characteristic polynomial, we first conduct long division

$$\begin{array}{r l}
 & -\lambda^2+\lambda+0.25 \\
\lambda-1 \,\big) & -\lambda^3+2\lambda^2-1.25\lambda+0.25 \\
 & -\lambda^3+\lambda^2 \\
\hline
 & \lambda^2-1.25\lambda+0.25 \\
 & \underline{\lambda^2-\lambda} \\
 & -0.25\lambda+0.25 \\
 & \underline{-0.25\lambda+0.25}
\end{array}$$

Hence

$$-\lambda^3+2\lambda^2-1.25\lambda+0.25=(\lambda-1)(-\lambda^2+\lambda+0.25)$$

To find zeroes of $-\lambda^2+\lambda+0.25$, we solve the quadratic equation,

$-\lambda^2+\lambda+0.25=0$, to give

$$\lambda=\frac{-(1)\pm\sqrt{(1)^2-(4)(-1)(0.25)}}{2(-1)}$$

$$=\frac{-1\pm\sqrt{0}}{-2}$$

$$=0.5,0.5$$

So $\lambda=0.5$ and $\lambda=0.5$ are the zeroes of

$-\lambda^2+\lambda+0.5$

Giving $-\lambda^2+\lambda+0.25=-(\lambda-0.5)(\lambda-0.5)$

Hence $-\lambda^3+2\lambda^2-1.25\lambda+0.25=0$ can be rewritten as

$-(\lambda-1)(\lambda-0.5)(\lambda-0.5)=0$ to give the roots as

$$\lambda=1,\ 0.5,\ 0.5$$

These are the three roots of the characteristic polynomial equation and hence the eigenvalues of matrix [A].

Note that there are eigenvalues that are repeated. Since there are only two distinct eigenvalues, there are only two eigen spaces. But, corresponding to $\lambda=0.5$ there should be two eigenvectors that form a basis for the eigen space corresponding to $\lambda=0.5$.

Given: $[(A-\lambda I)][X]=0$ then

$$\begin{bmatrix} 1.5-\lambda & 0 & 1 \\ -0.5 & 0.5-\lambda & -0.5 \\ -0.5 & 0 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{For } \lambda = 0.5,$$

$$\begin{bmatrix} 1 & 0 & 1 \\ -0.5 & 0 & -0.5 \\ -0.5 & 0 & -0.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system gives: $x_1 = -a, x_2 = b, x_3 = a$

$$\text{So} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a \\ b \\ a \end{bmatrix}$$

$$= \begin{bmatrix} -a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$$

$$= a\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So the vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ form a basis for the Eigen space for the eigenvalue $\lambda = 0.5$ and are the two eigenvectors corresponding to $\lambda = 0.5$.

For $\lambda = 1$,

$$\begin{bmatrix} 0.5 & 0 & 1 \\ -0.5 & -0.5 & -0.5 \\ -0.5 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system gives

$$x_1 = a, x_2 = -0.5a, x_3 = -0.5a$$

The eigenvector corresponding to $\lambda = 1$ is

$$\begin{bmatrix} a \\ -0.5a \\ -0.5a \end{bmatrix} = a\begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix}$$

Hence the vector

$$\begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix}$$

is a basis for the eigen space for the eigenvalue of $\lambda = 1$, and is the eigenvector corresponding to $\lambda = 1$.

5.5 THEOREMS OF EIGENVALUES AND EIGENVECTORS:

Theorem 1: If $[A]$ is a $n \times n$ triangular matrix – upper triangular, lower triangular or diagonal, the eigenvalues of $[A]$ are the diagonal entries of $[A]$.

Theorem 2: $\lambda = 0$ is an eigenvalue of $[A]$ if $[A]$ is a singular (noninvertible) matrix.

Theorem 3: $[A]$ and $[A]^{\mathrm{T}}$ have the same eigenvalues.

Theorem 4: Eigenvalues of a symmetric matrix are real.

Theorem 5: Eigenvectors of a symmetric matrix are orthogonal, but only for distinct eigenvalues.

Example :1

What are the eigenvalues of?

$$[A] = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 9 & 5 & 7.5 & 0 \\ 2 & 6 & 0 & -7.2 \end{bmatrix}$$

Solution: Since the matrix $[A]$ is a lower triangular matrix, the eigenvalues of $[A]$ are the diagonal elements of $[A]$. The eigenvalues are

$$\lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 7.5, \lambda_4 = -7.2$$

Example :2

One of the eigenvalues of

$$[A] = \begin{bmatrix} 5 & 6 & 2 \\ 3 & 5 & 9 \\ 2 & 1 & -7 \end{bmatrix} \text{is zero. Is } [A] \text{ invertible?}$$

Solution: $\lambda = 0$ is an eigenvalue of $[A]$, that implies $[A]$, is singular and is not invertible

Example :3

Given the eigenvalues of

$$[A]=\begin{bmatrix} 2 & -3.5 & 6 \\ 3.5 & 5 & 2 \\ 8 & 1 & 8.5 \end{bmatrix} \text{ are } \lambda_1=-1.547, \lambda_2=12.33, \lambda_3=4.711$$

What are the eigenvalues of $[B]$ if?

$$[B]=\begin{bmatrix} 2 & 3.5 & 8 \\ -3.5 & 5 & 1 \\ 6 & 2 & 8.5 \end{bmatrix}$$

Solution:

Since $[B]=[A]^T$, the eigenvalues of $[A]$ and $[B]$ are the same. Hence eigenvalues of $[B]$ also are

$$\lambda_1=-1.547, \lambda_2=12.33, \lambda_3=4.711$$

Example :4

Given the eigenvalues of

$$[A]=\begin{bmatrix} 2 & -3.5 & 6 \\ 3.5 & 5 & 2 \\ 8 & 1 & 8.5 \end{bmatrix}$$

Are $\lambda_1=-1.547, \lambda_2=12.33, \lambda_3=4.711$

Calculate the magnitude of the determinant of the matrix.

Solution:

Since $|\det[A]|=|\lambda_1||\lambda_2||\lambda_3|=|-1.547||12.33||4.711|=89.88$

One of the most common methods used for finding eigenvalues and eigenvectors is the power method. It is used to find the largest eigenvalue in an absolute sense. Note that if this largest eigenvalue is repeated, this method will not work. Also, this eigenvalue needs to be distinct. The method is as follows:

1.Assume a guess $[X^{(0)}]$ for the eigenvector in $[A][X]=\lambda[X]$

equation. One of the entries of $[X^{(0)}]$ needs to be unity.

2.Find

$$[Y^{(1)}]=[A][X^{(0)}]$$

3.Scale $[Y^{(1)}]$ so that the chosen unity component remains unity.

$$[Y^{(1)}]=\lambda^{(1)}[X^{(1)}]$$

4. Repeat steps (2) and (3) with

$[X] = [X^{(1)}]$ to get $[X^{(2)}]$.

5. Repeat the steps 2 and 3 until the value of the eigenvalue converges.

If E_s is the pre-specified percentage relative error tolerance to which you would like the answer to converge to, keep iterating until

$$\left|\frac{\lambda^{(i+1)} - \lambda^{(i)}}{\lambda^{(i+1)}}\right| \times 100 \le E_s$$

Where the left-hand side of the above inequality is the definition of absolute percentage relative approximate error, denoted generally by E_s A pre-specified percentage relative tolerance of $0.5 \times 10^{2-m}$ implies atleast m significant digits are current in your answer. When the system converges, the value of λ is the largest (in absolute value) eigen value of $[A]$

Example 5:

Using the power method, find the largest eigenvalue and the corresponding eigenvector of

$$[A] = \begin{bmatrix} 1.5 & 0 & 1 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & 0 & 0 \end{bmatrix}$$

Solution:

Assume

$$[X^{(0)}] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$[A][X^{(0)}] = \begin{bmatrix} 1.5 & 0 & 1 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$

$$Y^{(1)} = 2.5\begin{bmatrix} 1 \\ -0.2 \\ -0.2 \end{bmatrix}$$

$\lambda^{(1)} = 2.5$

We will choose the first element of $[X^{(0)}]$ to be unity.

$$[X^{(1)}] = \begin{bmatrix} 1 \\ -0.2 \\ -0.2 \end{bmatrix}$$

$$[A][X^{(1)}] = \begin{bmatrix} 1.5 & 0 & 1 \\ -0.5 & 0.5 & -0.5 \\ -0.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -0.2 \\ -0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.3 \\ -0.5 \\ -0.5 \end{bmatrix}$$

$$[X^{(2)}] = 1.3 \begin{bmatrix} 1 \\ -0.3846 \\ -0.3846 \end{bmatrix}$$

$$\lambda^{(2)} = 1.3$$

$$[X^{(2)}] = \begin{bmatrix} 1 \\ -0.3846 \\ -0.3846 \end{bmatrix}$$

The absolute relative approximate error in the eigenvalues is

$$|\varepsilon_a| = \left| \frac{\lambda^{(2)} - \lambda^{(1)}}{\lambda^{(2)}} \right| \times 100$$

$$= \left| \frac{1.3 - 1.5}{1.5} \right| \times 100$$

$$= 92.307\%$$

Conducting further iterations, the values of $\lambda^{(i)}$ and the corresponding eigenvectors is given in the table below

i	$\lambda^{(i)}$	$[X^{(i)}]$	
1	2.5	$\begin{bmatrix} 1 \\ -0.2 \\ -0.2 \end{bmatrix}$	
2	1.3	$\begin{bmatrix} 1 \\ -0.38462 \\ -0.38462 \end{bmatrix}$	
3	1.1154	$\begin{bmatrix} 1 \\ -0.44827 \\ -0.44827 \end{bmatrix}$	
4	1.0517	$\begin{bmatrix} 1 \\ -0.47541 \\ -0.47541 \end{bmatrix}$	$\lvert\varepsilon_a\rvert$ (%)
5	1.02459	$\begin{bmatrix} 1 \\ -0.48800 \\ -0.48800 \end{bmatrix}$	______
			92.307
			16.552

The exact value of the eigenvalue is $\lambda = 1$ and the corresponding eigenvector is

$$[X] = \begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix}$$

<u>5.6 Cayley Hamilton theorem:</u>

Every square matrix satisfies its own characteristic equation. Let A be a non-singular Matrix

i.e. $|\vec{A}|$ = 0 from the Cayley Hamilton theorem

We have $a_0 A^n + a_1 A^{n-1} + a_1 A^{n-2} + \dots\dots\dots + a_n I = 0$ …………(1)

Pre -Multiplying equation (1) by A^{-1} we get

$a_0 A^{n-1} + a_1 A^{n-2} + a_1 A^{n-3} + \dots\dots\dots + a_n A^{-1} = 0$ (since $A^{-1} I = A^{-1}$)

$a_n A^{-1} = -(a_0 A^{n-2} + a_1 A^{n-3} + \dots\dots\dots + a_{n-1} I)$

A^{-1} =1/a_n -($a_0 A^{n-2}$ +$a_1 A^{n-3}$ +.........+$a_{n-1} I$)

Example 1:

Verify that A=$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ satisfies its own characteristic equation & hence find A^4

To find the characteristic equation

Characteristic equation is $|A - I\lambda|$ =0

$\begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix}$ =0

$(1-\lambda)(-1-\lambda)$-4=0

-1$-\lambda$ +λ +λ^2-4 =0

λ^2-5=0

To find A^4 λ^2-5I=0

$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ -5$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$= $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Hence A Satisfies its own characteristic equation

Multiplying (A) by A^2, we get

A^4 -5A^2 = 0

A^4 =5A^2 =5$\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ =$\begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$

Example 2:

Use Cayley –Hamilton theorem to find the inverse of A= $\begin{pmatrix} 7 & 3 \\ 2 & 6 \end{pmatrix}$

To find the characteristic equation

Characteristic equation is $|A - I\lambda|$ =0

$\begin{vmatrix} 7-\lambda & 3 \\ 2 & 6-\lambda \end{vmatrix}$ =0

$(7-\lambda)(6-\lambda)$-6 =0

42 +λ^2-13λ -6 =0

λ^2-13λ -36 =0

To Find A^{-1}

By Cayley –Hamilton theorem we get

A^2-13A-36I =0

$A^{-1}A^2$-13AA^{-1}-36A^{-1} =0

A-13I+-36A^{-1} =0

A^{-1} =$\frac{1}{36}$ $(13\mathrm{I} - \mathrm{A})$

=$\frac{1}{36}\left(\begin{pmatrix}13 & 0\\ 0 & 13\end{pmatrix} - \begin{pmatrix}7 & 3\\ 2 & 6\end{pmatrix}\right)$

=$\frac{1}{36}\left(\begin{pmatrix}6 & -3\\ -2 & 7\end{pmatrix}\right)$

Example 3

Verify Cayley –Hamilton theorem for the matrix A=$\begin{bmatrix}8 & -8 & 2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{bmatrix}$

To find characteristic equation:

Let A =$\begin{bmatrix}8 & -8 & 2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{bmatrix}$the characteristic equation is λ^3-$a_1\lambda^2 + a_2\lambda$ -a_3 =0

Wherea_1 = sum of leading diagonal elements =8-3+1 = 6

a_2 = sum of the minors of the leading diagonal elements

=$\begin{vmatrix}-3 & -2\\ -4 & 1\end{vmatrix}$ +$\begin{vmatrix}8 & 2\\ 3 & 1\end{vmatrix} + \begin{vmatrix}8 & -8\\ 4 & -3\end{vmatrix}$

=-3-8+8-6-24+32

=-1

$a_3 = |A|$ =$\begin{vmatrix}8 & -8 & 2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{vmatrix}$

= 8(-3-8)+ 8(4+6)+2(-16+9)=-88+68-14

= -22

The characteristic equation is $\lambda^3 - 6\lambda^2 - \lambda$ +22 =0

Verification: To verify Cayley –Hamilton theorem we have to prove that $A^3 - 6A^2 - A$+ 22I =0

Now A^2 = $\begin{bmatrix}8 & -8 & 2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{bmatrix} + \begin{bmatrix}8 & -8 & 2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{bmatrix}$

$$= \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}$$

$$A^3 = A^2 * A = \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} * \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix}$$

$$A^3 - 6A^2 - A + 22I = \begin{bmatrix} 214 & -296 & 206 \\ 88 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix} -$$

$$\begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} - \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} + 22 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 4

Verify Cayley –Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

To find characteristic equation:

Let $A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$, The characteristic equation is $\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$

Where a_1 = sum of leading diagonal elements =1+1+1 = 3

a_2 = sum of the minors of the leading diagonal elements

$$= \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

= -1-1+1-3+1-0=-1

$$a_3 = |A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

= 1(1-1)+ 0(2+1)+3(-2-1)

= -9

The characteristic equation is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

Verification: To verify Cayley –Hamilton theorem we have to prove that $A^3 - 3A^2 - A+$ 9I =0

Now $$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$A^3 = A^2 * A = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix}$$

$A^3 - 3A^2 - A$+9I=

$$\begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix} - 3 \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley –Hamilton theorem is verified

5.7 Minimal Equations:

Let f(x) is a polynomial is the indeterminate x and A is a square matrix of order n. If f(x) = O, then we say that the polynomial f(x) annihilates the matrix A. Every matrix satisfies its characteristic equation and the characteristic polynomial of a matrix A is a non-zero polynomial, i.e a polynomial in which the coefficients of various terms are not all zero. Therefore, atleast the characteristic polynomial of A is a non-zero polynomial that annihilates A. Thus, the set of those non-zero polynomial which annihilate A is not empty.

5.7.1 Monic Polynomial:

A polynomial is x in which the coefficient of the highest power of x is unity is called a monic polynomial, e.g. $x^3 + 4x^2 - (3x) + 5$ is a monic polynomial of degree 3 over the field of real numbers. Among those non-zero polynomials which annihilates a matrix A, the polynomial which is monic and which is of the lowest degree is of special interest. It is called the minimal polynomial of the matrix A.

5.7.2 Minimal equation of a Matrix.

The monic polynomial of lowest degree that annihilates a matrix A is called the minimal polynomial of A. Also, if f(x) is the minimal polynomial of A, the equation f(x) =0 is called the minimal equation of the matrix A.

If A is of order n, then its characteristic polynomial is of degree n.

Since the characteristic polynomial of A annihilates A, therefore the minimal polynomial of A cannot be of degree greater than n. Its degree must b less than or equal to n.

Theorem 1: The minimal equation of a matrix is unique.

Let the minimal polynomial of a matrix A is of degree r. Then no non-zero polynomial of degree less than r on annihilates A. Let

f(x) = $x^r + a_1x^{r-1}+a_2x^{r-2} + \dots\dots\dots.. +a_{r-1}x + a_r$ and

g(x) =$x^r + b_1x^{r-1}+b_2x^{r-2} + \dots\dots\dots.. +b_{r-1}x + b_r$ be two minimal polynomials of A. Then both f(x) and g(x) annihilate A.

Therefore, we have

f(A) =O and g(A) = 0. These give

$A^r + a_1A^{r-1}+a_2A^{r-2} + \dots\dots\dots.. +a_{r-1}A + a_rI$ =0 …………..…(1) &

$A^r + b_1A^{r-1}+b_2A^{r-2} + \dots\dots\dots.. +b_{r-1}A + b_rI = 0$ …………..(2)

Subtracting (1) and (2), we get,

$(b_1 - a_1)A^{r-1}+((b_2 - a_2)A^{r-2} + \cdots \dots.(b_r = a_r)I$ =0 ………(3)

From (3) we see that the polynomial on L.H.S also annihilate A. Since the degree is less than r, therefore it must be a zero polynomial. This gives

$(b_1 - a_1) = 0, (b_1 - a_2) = 0, \dots\dots..(b_r - a_r) = 0.$

Thus $(b_1 = a_1)(b_1 = a_2), (b_r = a_r).$

$There$fore f(x) =g(x) and thus the minimal equation of A is unique

5.8 Quadratic Form:

A homogeneous polynomial of second degree in any number of variables is called a quadratic form

Note: homogeneous polynomial of second degree means each and every term in any expression should have degree two.

5.8.1 Matrix of the Quadratic form:

Examples:1

i) X_1^2+ 5X_1X_2 +X_2^2 is a quadratic form in the variables X_1 and X_2

ii) X_1^2+ 2X_1X_2 +X_2^2 +X_3^2 -3X_1X_3 +$X_2X_3$is a quadratic form in three variables X_1 and X_2. X_3

iii) X_1^2+ 2X_1X_2 +X_2^2 +X_3^2+-5X_1X_3 +5X_4^2 -2X_2X_3+$X_2X_4 + 5X_4X_3$ is a quadratic form in four variables X_1 ,X_2. X_3 and X_4

Note that the degree of each and every term in the above expression is two

A quadratic form in 3 variables X_1 and X_2. X_3 is given by

f(X_1 ,X_2,X_3) = $a_{11}X_1^2$ + a_{12} X_1 X_2 +a_{13} X_1 X_3 +a_{21} X_1 X_2 +$a_{22}X_2^2$ $+ a_{23}$ X_2 X_3 + a_{31} X_3 X_1 +a_{32} X_3 X_2 +$a_{33}X_3^2$

The quadratic form can be written by

f(X_1 ,X_2,X_3)=$\sum_{i=1}^{3} \sum_{j=1}^{3} a_{ij X_I X_J}$ =(X_1 ,X_2. X_3)$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

=X' AX

Where X=$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ and A is called the matrix of the Quadratic form

Example: 2

Write the matrix of the Quadratic form 2 X_1^2 + 2 X_1 X_2 -6 X_1 X_3 $+ 6$ X_2 X_3 - 2X_2^2 $+$ $4X_3^2$

Here a_{11}=2 a_{22}=-2 ,a_{33}=4 a_{12}=a_{21}=$\frac{2}{2}$ =1

a_{31}= 31=-$\frac{-6}{2} = -3$, a_{32}= a_{23}=a_{21}=$\frac{6}{2} = 3$

Hence the matrix of the form is $\begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}$

Example: 3

Write the Quadratic form corresponding to the following symmetric matrix is $\begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$

Solution: Quadratic form corresponding to the symmetric matrix A is

X' AX = $(X_1, X_2, X_3)\begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ Using matrix multiplication, we have

$= (0X_1 - X_2 + 2X_3 \quad X_1 + X_2 + 4X_3 \quad 2X_1 + 4X_2 + 3X_3)\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

$= (0X_1 - X_2 + 2X_3)X_1 + (-X_1 + X_2 + 4X_3)X_2 + (2X_1 + 4X_2 + 3X_3)X_3$

$= 0X_1^2 - 2X_1X_2 + 4X_1X_3 + 8X_2X_3 + X_2^2 + 3X_3^2$

Note

1. Rank of the symmetric matrix A is called the rank of the Quadratic form X' AX
2. If the Rank of A is r<n 9number of variables) then the Quadratic form is singular otherwise non-singular

   Transformation

Let X'AX be a quadratic form where A is the matrix of the quadratic form

Let X=PY be a non-singular linear transformation (P is non-singular) then we have

X'AX = (PY)' A PY

=P'Y' A PY=Y' (P'AP)Y

=Y'DY where D =(P'AP)

Let us choose P to be the matrix of a set of orthogonal eigenvectors of A. now the matrix P is orthogonal (since P'=$P^{-1}$and P'AP is a diagonal matrix D whose elements are the eigenvalues$\lambda_1, \lambda_2, \lambda_3$ of A

Here Y'DY is also a quadratic form in variable y_1, y_2, y_3 thus the quadratic form X'AX is reached to the canonical form

In other words, a quadratic form X'AX in 3 unknownx_1, x_2, x_3 can be reduced to the canonical form

$d_1 y_1{}^2 + d_2 y_2{}^2 + d_3 y_3{}^2$ Where y_1, y_2, y_3 are the new unknowns. Some of the coefficients d_1, d_2, d_3 may of course be zero.

Note:

1. If the matrix p is orthogonal the transformation X=PY is called an orthogonal transformation

2. The above method is applicable only when the eigenvectors of A are linearly independent and mutually orthogonal

<u>5.8.2 Theorem:</u> Fundamental theorem on quadratic forms. Any quadratic form may be reduced to canonical form by means of non-singular transformations

Proof: Let X'AX ……… (1) be a quadratic form of rank 3

Therefore, A is of rank 3. Then there exists a non-singular matrix P, such that

(P'AP) $=\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$ where D is exists a non-singular matrix of order 3

Apply the non-singular transformations X=PY in (1) where

$X=\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ and $Y=\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$ we get

$X'AX = (PY)'A(PY)$

$= YP'APY$

$= Y'(P'AP)Y$

$= Y'\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} Y$ by (2)

$= (Y_1, Y_2. Y_3)\begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = d_1 y_1{}^2 + d_2 y_2{}^2 + d_3 y_3{}^2$ which is the canonical form of the given quadratic form.

Example 1

Reduce the quadratic form $2X_1^2 + 2\ X_1 X_2 - 2\ X_1 X_3 - 4\ X_2 X_3 + X_2^2 + X_3^2$ to canonical form through an orthogonal transformation.

Solution: The given quadratic form is X'AX, where $X=\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

and $X' = (X_1 \quad X_2 \quad X_3)$

Step: 1 To find the matrix of the quadratic form:

The matrix of the quadratic form is

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$$

Step: 2 To find characteristic equation:

The characteristic equation is $\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$

Where a_1 = sum of leading diagonal elements

$= 2+1+1 = 4$

a_2 = sum of the minors of the leading diagonal elements

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$$

$= -1-4+2-1+2-1$

$= -1$

$$a_3 = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix}$$

$= 2(1-4)-1(1-2)-1(-2+1)$

$= -4$

The characteristic equation is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

Step: 3 To find eigenvalues:

$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

When $\lambda = 1$, $1-4-1+4 = 0$

Therefore $\lambda = 1$ is a root

$\lambda^2 - 3\lambda - 4 = 0$ $\quad \lambda = \frac{-3 \pm \sqrt{9+16}}{2} = \frac{3+5}{2} = 4$ or -1

Eigen values are $\lambda = 1, -1, 4$

Step : 4 to find eigenvectors:

The eigenvectors X $= (X_1 \quad X_2 \quad X_3)$ are given by

$(2-\lambda)X_1 + X_2 - X_3 = 0$

$X_1 + (1-\lambda)X_2 - 2X_3 = 0$

$-X_1 - 2X_2 + (1-\lambda)X_3 = 0$

Case (i): When $\lambda = -1$, the eigenvector is given by

$3X_1 + X_2 - X_3$ =0

$X_1 + 2X_2 - 2X_3$ =0

$-X_1 - 2X_2 + 2X_3$ =0

Taking first two equations and solving, we get

X_1		X_2	X_3
1	-1	3	1
2	-2	1	2

$\frac{X_1}{-2+2} = \frac{X_2}{-1+6} = \frac{X_3}{6-1}$ =k

$\frac{X_1}{0} = \frac{X_2}{5} = \frac{X_3}{5}$ =k

$X_1 = 0k$: $X_2 = 5k$

X_3 =5k (taking k= $\frac{1}{5}$) i.e the eigenvector is (0,1,1) and its normalized form is $(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$

Case (ii): When λ =-1, the eigenvector is given by

$X_1 + X_2 - X_3$ =0

$X_1 + 0X_2 - 2X_3$ =0

$-X_1 - 2X_2 + 0X_3$ =0 considering the first equations, we have

X_1		X_2	X_3
1	-1	1	1
0	-2	1	0

$\frac{X_1}{-2-0} = \frac{X_2}{-1+2} = \frac{X_3}{0-1}$ =k

$X_1 = -2k$: X_2 =k:

X_3 =- k (taking k= -1) i.e. the eigenvector is (2, -1,1) and its normalized form is $(\frac{2}{\sqrt{6}},\frac{-1}{\sqrt{6}},\frac{1}{\sqrt{6}})$

Case (iii): When λ =4, the eigenvector is given by

$-2X_1 + X_2 - X_3$ =0

$X_1 - 3X_2 - 2X_3$ =0

$-X_1 - 2X_2 - 3X_3$ =0 considering the first equations, we have

X_1		X_2	X_3
1	-1	-2	1
-3	-2	1	-3

$\frac{X_1}{-2-3} = \frac{X_2}{-1-4} = \frac{X_3}{6-1}$ =k

$X_1 = -5k$: $X_2 = -5k$

$X_3 = -5k$ (taking k=$\frac{-1}{5}$) i.e. the eigenvector is (1,1, -1) and its normalized form is $(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$

Step: 5 to find modal matrix:

The normalized modal matrix is

P=$\begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$,P' =$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$ Let X =PY……….(2) be the orthogonal transformation substituting (2) in (1), we get

X'AX =$(PY)'$A(PY) =YP'APY

Now Y' (P'AP) Y = $(Y_1, Y_2. Y_3)\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$

= $-y_1{}^2 + y_2{}^2 + 4y_3{}^2$ which is the required canonical form of the given quadratic form.

Example: 2

Reduce the quadratic form X_1^2 - 2 $X_1 X_2$ -2 $X_1 X_3 + 2X_2 X_3$ +$2X_2^2$ + X_3^2 to canonical form through an orthogonal transformation

Solution: The given quadratic form is X'AX where

$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ and X' $= \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}$

Step: 1 To find the matrix of the quadratic form:

The matrix of the quadratic form is

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Step: 2 To find characteristic equation:

The characteristic equation is $\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$

Where a_1 = Sum of leading diagonal elements =1+2+1 = 4

$a_{2=}$ = Sum of the minors of the leading diagonal elements

$= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$ =2-1+1+2-1=3

$$a_{3=}|A| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

= 1(2-1) +1(-1-0) +0 = 0

The Characteristic equation is $\lambda^3 - 4\lambda^2 - 3\lambda = 0$

Step: 3 To find eigenvalues:

$\lambda^3 - 4\lambda^2 - 3\lambda = 0$, $\lambda(\lambda^2 - 4\lambda^1 + 3) = 0$

When $\lambda = 0 (\lambda^2 - 4\lambda^1 + 3) = 0$, $\lambda = \frac{-4 \pm \sqrt{16-12}}{2} = \frac{4+2}{2}$ =3 or 1

Eigen values are λ = 0,1,3

Step: 4 to find eigenvectors:

The eigenvectors X $= \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}$ are given by

$(1-\lambda) X_1 - X_2 - 0\, X_3 = 0$

$-X_1 + (2-\lambda) X_2 + X_3 = 0$

$0X_1 + X_2 + (1-\lambda) X_3 = 0$

Case (i): When λ =0, the eigenvector is given by

$X_1 - X_2 + 0\, X_3 = 0$

$-X_1 + 2\,X_2 + X_3$ =0

$0X_1 + X_2 + X_3$ =0

Taking first two equations and solving, we get

X_1	X_2	X_3	
-1	0	1	-1
2	1	-1	2

$\frac{X_1}{-1-0} = \frac{X_2}{0-1} = \frac{X_3}{2-1}$ =k

$\frac{X_1}{-1} = \frac{X_2}{-1} = \frac{X_3}{1}$ =k

$X_1 = -k$: X_2 =k:

X_3 =k (taking k=-1) i.e. the eigenvector is (1,1,-1) and its normalized form is $(\frac{1}{\sqrt{3}}, \frac{1}{3}, \frac{-1}{\sqrt{3}})$

Case (ii): When λ =1, the eigenvector is given by

$0X_1 - X_2 - 0\,X_3$ = 0

$-X_1 + X_2 + X_3$ = 0

$0X_1 + X_2 + 0X_3$ =0 considering the first equations, we get

X_3 = 0

$X_1 = X_3$

Therefore the eigenvector is (1, 0, 1) and its normalized form is $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

Case (iii): When λ =3, the eigenvector is given by

$-2X_1 - X_2 + 0\,X_3$ =0

$-X_1 - X_2 + X_3$ =0

$0X_1 + X_2 - 2X_3$ =0 considering the first equations and solving we get

X_1	X_2	X_3	
-1	0	-2	-1
-1	1	-1	-1

$\frac{x_1}{-1-0} = \frac{x_2}{0+2} = \frac{x_3}{2-1} = k$

$X_1 = -k$: $X_2 = 2k$, $X_3 = k$ (taking k=-1) i.e. the eigenvector is (1,-2,-1) and its normalized form is $(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{-1}{\sqrt{6}})$

Step: 5 to find modal matrix:

The normalized modal matrix is

$$P=\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{pmatrix}, \quad P' = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

Let X =PY………(2) be the orthogonal transformation substituting (2) in (1),

we get

$$X'AX = (PY)'A(PY)$$
$$= YP'APY$$

Now $Y'(P'AP)Y = (Y_1, Y_2, Y_3)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$

$= {y_2}^2 + 3{y_3}^2$ which is the required canonical form of the given quadratic form.

Example 3

Reduce the quadratic form X_1^2 - $+2\,X_2\,X_3$ to canonical form by means of an orthogonal transformation. Determine its nature

Solution:

The given quadratic form is X'AX $= X_1^2$ - $+2\,X_2\,X_3$

Step: 1 To find the matrix of the quadratic form:

The matrix of the quadratic form is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Step: 2 To find characteristic equation:

The characteristic equation is $\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$

Where a_1 = Sum of leading diagonal elements

=1+0+0 = 1

a_2 = Sum of the minors of the leading diagonal elements

$=\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 1$

$a_3 = |A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1$

The characteristic equation is $\lambda^3 - \lambda^2 - \lambda = -1$

Step :3 To find eigenvalues: $\lambda^3 - \lambda^2 - \lambda = -1$

When $\lambda = 1$ 1-1-1+1 =0 therefore $\lambda = 1$ is a root

$\lambda^2 = 1 = \pm 1$

Eigen values are $\lambda = 1, 1, -1$

Step: 4 to find eigenvectors:

The eigenvectors X $= (X_1 \quad X_2 \quad X_3)$ are given by

$(1-\lambda) X_1 + 0\, X_2 + 0\, X_3 = 0$

$0\, X_1 - \lambda\, X_2 + X_3 = 0$

$0 X_1 + X_2 + -\lambda X_3 = 0$

Case (i): When $\lambda = -1$, the eigenvector is given by

$2X_1 = 0$

$2X_2 + X_3 = 0$

$X_2 + X_3 = 0 \Rightarrow X_2 = -X_3$ put $X_3 = k,\ X_2 = -k,\ X_1 = 0$

Therefore, the eigenvector is X $= \begin{pmatrix} 0 \\ -k \\ k \end{pmatrix}$ the simplest eigen vector is

$X_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

Case (ii): When $\lambda = 1$, the eigenvector is given by

$0X_1 = 0 \Rightarrow X_1$ takes any value

$-X_2 + X_3 = 0$

$X_2 - X_3 = 0$ …………………………………………………………..(1)

$X_2 = X_3$ Put $X_2 = k,\ X_3 = k$

$X_1 = k_1 = 0$

The simplest eigen vector is

$X_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ taking k=-1

Case (iii): Let $X_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the third eigenvector which is the orthogonal to X_2

$-x + y + z = 0$ ……………………………………..……(A)

Also, we have y-z =0 (therefore X_3 satisfies (1) ……………..……(B)

(B) $\Rightarrow$ y= z ……………………………………………….…(C)

X=2z (sub(C) in (1))

Take z=1, y=1, x=2

The eigenvector $X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Now we have the following 3 eigenvectors

$$X_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, X_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Step: 5 to find modal matrix:

The normalized modal matrix is

$$P = \begin{pmatrix} 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{pmatrix}, P' = \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{6} \end{pmatrix}$$

Step :6 To find $P'AP$

Let X =PY………. (2) be the orthogonal transformation substituting (2) in (1), we get

X'AX $=(PY)'$A(PY) =YP'APY

Now $Y'(P'AP)Y = (Y_1, Y_2, Y_3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$

$= {y_1}^2 + {y_2}^2 - {y_3}^2$ which is the required canonical form of the given quadratic form. Since the eigenvalues are 1,1,-1.Hence one eigenvalue is negative and two eigenvalues are positive. Hence the quadratic form is indefinite.

5.9 Index and Signature of the real Quadratic Form:

Let X'AX be the given quadratic form in the variablesx_1,x_2…………x_n

i.e X'AX = $d_1{x_1}^2$ +$d_2{x_2}^2 + d_3{y_3}^2$+……… +$d_n{x_n}^2$ ……………..(1)

Let the rank of A be r. Then X'AX consists only 'r' terms

The number of positive terms in (1) is called the index of the quadratic form and it is denoted by 's'. The difference between the number of positive terms and the negative terms Is called the signature of the quadratic form (i.e)

signature= $(number\ of\ posotive\ terms) - (number\ of\ negative\ terms)$

= s- (rank of A-s) = s-(r-s)

Therefore s =2s-r

Where s - number of positive terms

r-rank of A

Examples: 1

Find the index and signature of the quadratic form 3 X_1^2 - 2 $X_1\,X_2$ +2 $X_1\,X_3 - 2\,X_2\,X_3$ +$5X_2^2\ + 3X_3^2$

Solutions: The matrix of the quadratic form is A=$\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

The rank of A is 3

The canonical form of the above quadratic form is 2 y_1^2 +$3y_2^2\ +\ 6y_3^2$

Now Index(s) = Number of positive items=3

Rank(r) =3

Therefore signature =2s-r = 6-3 =3

5.9.1 Classification of Quadratic Form:

Let X'AX be the given real quadratic form where 'A' is the matrix of the quadratic form.

Let the eigenvalues of A be,λ_1,λ_2,3. Now the quadratic form X'AX is said to be

a) Positive definite if all the eigenvalues λ_1,λ_2,λ3 are positive

b) Negative definite if all the eigenvalues $\lambda_1, \lambda_2, \lambda 3$ are negative

c) Positive semi definite if at least one eigenvalues is zero and remaining are positive

d) Negative semi definite if at least one eigenvalues is zero and remaining are negative

e) Indefinite if some eigenvalues are positive and some eigenvalues are negative

Example : 1

Discuss the nature of the quadratic form

$$10X_1^2 - 4X_1X_2 - 10X_1X_3 + 6X_2X_3 + 2X_2^2 + 5X_3^2$$

The matrix of the quadratic form is $A = \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{pmatrix}$

The eigenvalues of A are 0,3,14. Here one eigenvalue is 0 and the remaining are positive.

Hence the given quadratic form is positive semi definite

Example: 2

Discuss the nature of the quadratic form

$3X_1^2 - 2X_1X_2 - 6X_1X_3 - 6X_2X_3 + 3X_2^2 - 5X_3^2$

The matrix of the quadratic form is $A = \begin{pmatrix} -3 & -1 & -3 \\ -1 & 3 & -3 \\ -3 & -3 & -5 \end{pmatrix}$

The eigenvalues of A are 4, -1,-8. Here we have positive and negative.

Hence the given quadratic form is indefinite.

5.9.2 Null Space & Nullity of a Matrix:

<u>Definition:</u> The subspace generated be the vectors X such that AX= O is called the column null space of the m x n matrix A and its dimension n-r called the column nullity of the matrix. Thus

Rank+ column nullity = No. columns

Note: Similarly, the subspace of the solution of

Y A=O is called the row null space and its dimension m -r is the row nullity of the matrix so that Rank+ row nullity = Number of rows

5.9.3 Reduction of a real quadratic form:

Theorem 1: If A be any n-rowed real symmetric matrix of rank r, then there exists a real non-singular matrix P such that, P'AP = diag [1,1,...,1,-1,-1,....,-1,0,...0]

So, that 1, appears p times and, -1, appears r-p times.

Proof: A is a real symmetric matrix of rank r. Therefore, there exists a non-singular real matrix Q such that Q'AQ is a diagonal matrix D with precisely r non-zero diagonal elements.

Let Q'AQ = D= diag [$\lambda_1, \lambda_2, \dots\dots\dots \lambda_r, 0, \dots\dots 0$].

Suppose that p of the non-zero diagonal elements are positive. Then r-p are negative.

Since in a diagonal matrix the positions of the diagonal elements occurring in i^{th} & j^{th} rows are interchanged by applying the congruent operation $R_i <=> R_j$,

$C_i <=> C_j$, . Therefore, without any loss of generality we can take $\lambda_1, \lambda_2, \dots\dots\dots \lambda_p$ to be positive and $\lambda_1, \lambda_2, \dots\dots\dots \lambda_r$ to be negative.

Let S be the n x n (real) diagonal matrix with diagonal elements.

If we take P= QS, then P is also real non-singular matrix and we have

P'AP = (QS)' A (QS) = S'Q'AQS = S'DS = SDS

= diag [1,1,...,1,-1,-1,....,-1,0,...0]

So, that 1 and -1 appear p and r-p times respectively.

Corollary: If X'AX is a real quadratic form of rank r in n variables, then there exists a real non-singular liner transformation X= PY which transforms X'AX to the form

$Y'P'APY = y_1^2 + \dots\dots\dots\dots\dots + y_p^2 - y_{p+1}^2 - \dots\dots\dots\dots - y_r$

5.9.4 Canonical or Normal form of a real quadratic form Definition:

If X'AX is a real quadratic form in a variable, then there exists a real non-singular liner transformation X= PY which transforms X'AX to the form

$Y'P'APY = y_1^2 + \dots\dots\dots\dots\dots + y_p^2 - y_{p+1}^2 - \dots\dots\dots\dots - y_r^{\ 2}$

In the new form the given quadratic form has been expressed as a sum and difference of the squares of the new variables. This latter expression is called the canonical form or normal form of the given quadratic form.

If Ǿ = X'AX is a real quadratic form of rank r, then A is a matrix of rank r. If the real non-singular liner transformation X =PY reduces Ǿ to normal form, then P'AP is a diagonal matrix having 1 and -1 as its non-zero diagonal elements.

Since P'AP is also of rank r, therefore it will have precisely r non-zero diagonal elements. Thus, the number of terms in each normal form of a given real quadratic form is the same. Now we shall prove that the number of positive terms in any two normal reductions of a real quadratic form is the same.

Theorem 1. The number of positive terms in any two normal reductions of a real quadratic form is the same.

Proof: Let Ǿ = X'AX is a real quadratic form of rank r in n variables. Suppose the real non-singular linear transformations

X=PY and X=QZ

Transform Ǿ to the normal forms,

$y_1^2 + \dots\dots + y_p^2 - y_{p+1}^2 - \dots\dots - y_r^2$ ---------- (1)

$z_1^2 + \dots\dots + z_q^2 - z_{q+1}^2 - \dots\dots - z_r^2$ ----------- (2)

and respectively.

To prove that p=q.

Let p<q. Obviously $y_1, \dots\dots y_n$, $z_1, \dots\dots z_n$, are linear homogeneous functions of, $x_1, \dots\dots x_n$

Since q>p, therefore q-p>0. So, n-(q-p) is less than n. Therefore (n-q) +p is less than n.

Now $y_1 = 0$, $y_2 = 0, \dots\dots y_p = 0$, $z_{q+1} = 0$, $z_{q+2} = 0, \dots\dots z_n = 0$ are (n-q)+p linear homogeneous equations in n unknowns n, therefore these equations must possess a non-zero solutions.

Let $x_1 = a_1, \dots\dots x_n = a_n$ be a non-zero solution of these equations and let $X_1 = [a_1, \dots\dots a_n]'$. Let $Y = [b_1, \dots\dots b_n]' = Y_1$ and $Z = [c_1, \dots\dots cn]'$ when $X = X_1$. Then $b_1 = 0, \dots\dots b_p = 0$ and $c_{q+1} = 0$, $c_n = 0$. Putting $Y = [b_1, \dots\dots b_n]'$ in (1) and $Z = [c_1, \dots\dots cn]'$ in (2), we get two values of Ǿ when $X = X_1$.

These must be equal. Therefore, we have

$-b^2_{p+1} - \dots\dots - b^2_r = c_1^2 + \dots\dots + c_q^2$

$\Rightarrow b_{p+1} = 0, \dots\dots b_r = 0$

$\Rightarrow Y_1 = 0$

$\Rightarrow P^{-1} X_1 = 0$ [$X_1 = PY_1$]

$\Rightarrow x_1 = 0$

Which is a contradiction since X_1 is a non-zero vector.

Thus, we cannot have p<q. Similarly, we cannot have q<p. Hence, we must have p=q.

Corollary. The number of negative terms in any two normal reductions of a real quadratic form is the same. Also, the excess of the number of positive terms over the number of negative terms in any two normal reductions of a real quadratic form is the same.

5.9.5 Signature and index of a real quadratic form.

Definition: Let $y_1^2 + \dots\dots + y_p^2 - y_{p+1}^2 - \dots\dots - y_r^2$ be a nominal form of a real quadratic form X'AX of rank r. The number p of positive terms in a normal form of X'AX is called the index of the quadratic form. The excess of the number of positive terms over the number of negative terms in a normal form of X'AX i.e..., p-(r-p) =2p-r is called the signature of the quadratic form and is usually denoted by s.

Thus s= 2p-r.

Theorem 1: Two real quadratic forms in n variables are real equivalent if and only if they have the same rank and index (or signature).

Proof: Suppose X'AX and Y'BY are two real quadratic forms in the same number of variables.

Let us first assume that the two forms are equivalent. Then there exists a real non-singular linear transformation X=PY which transforms X'AX and Y'BY i.e. B= P'AP.

Now suppose the real non-singular linear transformation Y =QZ transforms Y'BY to normal form Z'CZ. Then C=Q'BQ. Since P and Q are real non-singular matrices, therefore PQ is also a real non-singular matrix. The linear transformation X= (PQ)Z will transform X'AX to the form

(PQZ)' A(PQZ) = Z'Q'P' APQZ =Z'Q'BQZ =Z'CZ.

Thus, the two given quadratic forms have a common normal form. Hence, they have the same rank and the same index (or signature).

Conversely, suppose that the two forms have the same rank r and the same signature s. Then they have the same index p where 2p-r =s. So, they can be reduced to the same normal form

Z'CZ = $z_1^2 + \ldots\ldots\ldots\ldots\ldots + z_p^2 - z_{p+1}^2 - \ldots\ldots\ldots\ldots - z_r^2$

be real non-singular linear transformations, say, X=PZ and Y=QZ respectively. Then P'AP = C and Q'BQ =C.

Therefore Q'BQ = P'AP. This gives B= $(Q')^{-1}$ P'APQ^{-1} =$(Q^{-1})'$ P'APQ^{-1}=$(PQ^{-1})'$A(PQ^{-1}). Therefore the real non-singular transformation X=(PQ^{-1}) Y transforms X'AX to Y'BY. Hence the two given quadratic forms are real equivalent.

5.9.6 Reduction of a real quadratic form in the complex field.

Theorem 1. If A be any n-rowed real symmetric matrix of rank r, there exists a non-singular matrix P whose elements may be any complex numbers such that

P'AP= diag[1, 1,...., 1, 0,....,0] where 1, appears r times.

Proof: A is a real symmetric matrix of rank r. Therefore there exists a non-singular real matrix Q such that Q'AQ is a diagonal matrix D with precisely r non-zero diagonal elements. Let

Q'AQ=D=diag. [λ_1,..., λ_r, 0,..., 0].

The real numbers λ_1,..., λ_r may be positive or negative or both.

Let S be the nxn (complex) diagonal matrix with diagonal elements $\frac{1}{\sqrt{x_1}}, \ldots\ldots \frac{1}{\sqrt{x_r}}, , , , , , , , , , , 1, \ldots\ldots..1$ then S= Diag[$\frac{1}{\sqrt{\mu_1}}, \ldots\ldots \frac{1}{\sqrt{\mu_r}}, 1, \ldots\ldots..1$] is a complex non-singular diagonal matrix and S'=S.

If we take P=QS, then P is also a complex non-singular matrix and we have P'AP=(QS)'A (QS)=S'Q'AQS=S'DS=SDS=diag. [1, 1,...., 1, 0,....,0] so that 1 appears r times. Hence the result.

Corollary 1: Every real quadratic form X'AX is a complex equivalent to the form $z_1^2 + z_2^2 + \ldots.\ z_r^2$ where r is the rank of A.

Corollary 2: Two real quadratic forms in n variables are complex equivalent if and only if they have the same rank.

5.9.6 Orthogonal reduction of a real quadratic form.

Theorem 1. If $\varphi = X^{'}AX$ be a real quadratic form of rank r in n variables, then there exists a real orthogonal transformation X=PY which transforms φ to the form

$$\lambda_1 y_1^2 + ... + \lambda_r y_r^2,$$

where $\lambda_1,..., \lambda_r$ are the, r , non-zero eigen values of A, n-r eigen values of A being equal to zero.

Proof: Since A is real asymmetric matrix, therefore there exists a real orthogonal matrix P, such that

$$P^{-1}AP=D,$$

Where D is a diagonal matrix whose diagonal elements are the eigen values of A.

Since A is of rank r, therefore $P^{-1}AP=D$ is also a rank of r. So, D has precisely r non-zero diagonal elements. Consequently, A has exactly r non-zero eigenvalues, the remaining n-r eigenvalues of A being zero. Let D= diag. $[\lambda_1,..., \lambda_r, 0,..., 0]$.

Since $P^{-1}=P^{'}$, therefore $P^{-1}AP=D \rightarrow P^{'}AP=D \rightarrow$ A is congruent to D.

Now consider the real orthogonal transformation X=PY. We have $X^{'}AX=(PY)^{'}A\ (PY) = Y^{'}P^{'}APY = Y^{'}PY= \lambda_1 Y_1^2 + ... + \lambda_r Y_r^2$.

Hence the result.

5.9.7 Sylvester's law of inertia:

The signature of a real quadratic form is invariants for all normal reductions

Theorem 1: Sylvester's Law of Inertia. The signature of a real quadratic form is invariant for all normal reductions.

The number of positive terms in any two normal reductions of a real quadratic form is the same.

Proof: Let Ǿ = X'AX is a real quadratic form of rank r in n variables. Suppose the real non-singular linear transformations

X=PY and X=QZ

Transform Ǿ to the normal forms,

$y_1^2 + ... + y_p^2 - y_{p+1}^2 - .... - y_r^2$ ---------- (1)

$z_1^2 + ... + z_q^2 - z_{q+1}^2 - .... - z_r^2$ ----------- (2) and respectively.

To prove that p=q.

Let p<q. Obviously $y_1, ... y_n, z_1, ... z_n,$ are linear homogeneous functions of, $x_1, ... x_n$

Since q>p, therefore q-p>0. So, n-(q-p) is less than n. Therefore (n-q) +p is less than n.

Now $y_1 = 0, y_2 = 0, \ldots\ldots\ldots\ldots y_p = 0$,

$z_{q+1} = 0, z_{q+2} = 0, \ldots\ldots\ldots z_n = 0$ are (n-q)+p linear homogeneous equations in n unknowns, therefore these equations must possess a non-zero solutions.

Let $x_1 = a_1, \ldots\ldots\ldots\ldots x_n = a_n$ be a non-zero solution of these equations and

let $X_1 = [a_1, \ldots\ldots\ldots\ldots a_n]'$. Let $Y = [b_1, \ldots\ldots\ldots\ldots b_n]' = Y_1$

and $Z = [c_1, \ldots\ldots\ldots\ldots cn]'$ when $X = X_1$.

Then $b_1 = 0, \ldots\ldots\ldots b_p = 0$ and $c_{q+1} = 0, c_n = 0$. Putting

$Y = [b_1, \ldots\ldots\ldots\ldots b_n]'$ in (1) and $Z = [c_1, \ldots\ldots\ldots\ldots cn]'$ in (2), we get two values of $\acute{\O}$ when $X = X_1$.

These must be equal. Therefore, we have

$-b^2_{p+1} - \ldots\ldots\ldots\ldots - b^2_r = c_1^2 + \ldots\ldots\ldots + c_q^2$

$\Rightarrow b_{p+1} = 0, \ldots\ldots\ldots\ldots b_r = 0$

$\Rightarrow Y_1 = 0$

$\Rightarrow P^{-1} X_1 = 0 \; [X_1 = PY_1]$

$\Rightarrow x_1 = 0$

Which is a contradiction since X_1 is a non-zero vector.

Thus, we cannot have p<q. Similarly, we cannot have q<p. Hence, we must have p=q.

Theorem 2: If A and b are two n-rowed square matrices, then max{(v(A).

v(B) $\leq v(AB)$)}$\leq v(A + v(B)$ Here $v(A)$, $v(B), v(AB)$ denote the nullities of the square matrices A , B, AB respectively

We have already proved that

$\rho(A) \leq \rho(B) - n \;\; \leq \rho(AB)$)}$\leq \min\{\rho(A).\rho(B)\}$ …………(1)

Now $\rho(A) \leq n - v(A), \rho(A) \leq n - v(B), \rho(AB) \;\; \leq n - v(AB)$

Substituting these values in {1}, we prove the theorem

Note: The theorem was found by Sylvester in 1984

Course Material Prepared by
RAVICHANDRAN. M,
Assistant Professor, Department of Statistics,
PSG College of Arts & Science, Coimbatore - 14.